Commutators in the Rubik's Cube Group

Puzzle Enthusiast

Abstract. We show that every position of the Rubik's Cube generated by an even number of quarter turns can be solved by a single commutator. In other words, every element of the commutator subgroup of the Rubik's Cube group is itself a commutator. A generalization of the main result to the $n{\times}n{\times}n$ Rubik's Cube is also sketched.

1. INTRODUCTION Both mathematicians [**4, 6, 8**] and *cubers* [**3, 12**] (Rubik's Cube enthusiasts) alike have puzzled over the following question:

"Given a finite group, is every element of its commutator subgroup equal to a commutator?"

In a group, a *commutator* is a product of the form $[x, y] = xyx^{-1}y^{-1}$. To a mathematician, x and y are usually two elements of the group, but a cuber thinks of x and y as sequences of face turns of the Rubik's Cube. More formally, the face turns form a generating set of the Rubik's Cube group, and x and y are words on that set. Since each solvable position, by definition, can be expressed as a sequence of turns, these two notions are equivalent.

Commutators play perhaps an unexpected role in modern *speedcubing*, where cubers attempt to solve the Rubik's Cube as quickly as possible. They feature prominently in both blindfolded and fewest moves solving, where they can be used to permute a small subset of the pieces in only a few moves. In an arbitrary scrambled position, how much can a single commutator fix? The *commutator subgroup* is the subgroup generated by all of the commutators, and in the Rubik's Cube group, this subgroup consists of all positions which are solvable using an even number of quarter turns. We show that each of these positions, and hence half of all solvable positions, is equal to some commutator.

Our approach combines a known method of finding commutators in the symmetric group with a partial characterization of the conjugacy classes of the Rubik's Cube group. Speedcubers might be disappointed to know that this solution is unlikely to help with any human solving methods, falling under the aforementioned "mathematical" viewpoint of a commutator. Nevertheless, our results are expressed in the same way as a cuber would: our commutators *solve* a particular position, that is, the commutator is equal to the position's inverse, and in our examples, we write out each term of a commutator using sequences of turns. The optimal sequences found here were discovered using Kociemba's wonderful Cube Explorer program [**7**].

2. NOTATION AND TERMINOLOGY A *permutation* on a set X, whose elements we call *symbols*, is a bijection $P : X \to X$. A permutation P *fixes* a symbol $x \in X$ if $P(x) = x$. Otherwise, P *moves* x. An *ℓ-cycle* is a permutation whose only nontrivial cycle has length ℓ. A *transposition* is a 2-cycle, i.e., a permutation that just swaps two symbols. Every permutation on a finite set X can be decomposed into a product of transpositions, and one way of defining the *parity* of a permutation is the parity of the number of terms in that decomposition.

The group of all permutations on X, where multiplication is defined by function composition, is known as the *symmetric group* on X, and S_n denotes the symmetric

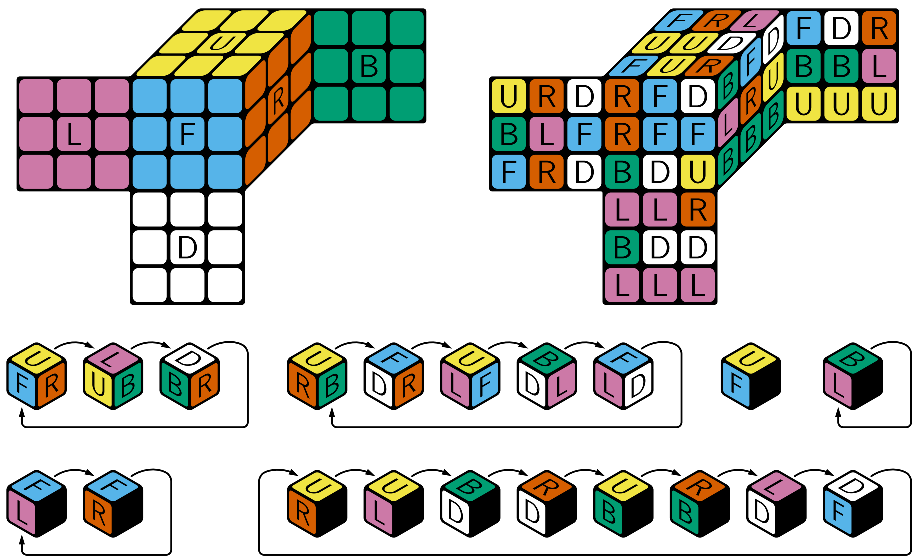


Figure 1. A solved cube, and a scrambled cube with its cycle decomposition.

group on a set of n symbols. The *alternating group* A_n is the subgroup of S_n consisting of all the even permutations. The Rubik's Cube is a classic real-life example of a group, since every position can be thought of as a permutation on the stickers. In this paper, group multiplication (especially in a sequence of turns) is read from left to right.

The Rubik's Cube is a mechanical puzzle that appears to be built out of 27 smaller *pieces*. Figure 1 shows the solved state as a partially unfolded net of a cube. The standard names of the faces are U(p), F(ront), R(ight), B(ack), L(eft), and D(own), and in this paper, those faces have yellow, blue, red, green, purple, and white stickers, respectively. Each face can be turned 90 degrees, rotating the $1\times3\times3$ layer of pieces incident with that face. If X is one of the faces, we use X, X2, and X' to denote clockwise 90 degree, 180 degree, and counter-clockwise 90 degree turns, respectively. A position is said to be *even* if it can be reached from the solved state in an even number of quarter turns. Otherwise, the position is *odd*. Each quarter turn applies an odd permutation to the stickers of the Rubik's Cube, so the parity of a position is well-defined.

The hidden inner piece is the core that holds the mechanism together, and the *center* pieces in the middle of each face can only rotate in place. The remaining *corner* and *edge* pieces can be shuffled around the puzzle, forming two orbits of pieces and stickers. Until Section 9, when we talk about "pieces," we mean just the corners and edges. Those pieces are given names that list out its incident faces, in counter-clockwise order. The letter that comes first often indicates the orientation of the piece, i.e., which sticker is "facing up."

Recall that a permutation can be described by its cycle decomposition, and the *cycle type* of a permutation is the sequence of cycle lengths, sorted in decreasing order. Though the stickers of the Rubik's Cube are the symbols being permuted, a cycle decomposition of stickers is cumbersome to parse because it contains redundant information. For example, the destination of one corner sticker determines those of the other two stickers on that piece. A more compact representation is due to Singmaster [**11**], which tracks permutations of *pieces* and their orientations. We illustrate this notation on the scrambled position in Figure 1, which can be generated from the solved state by the sequence:

B2 F' U' F R' U F2 D2 R' B2 R' D R2 B2 U' B' D2.

Consider the UFR corner piece in the scrambled position. It has been sent to LUB, where the U sticker of UFR has been mapped to the L sticker of LUB. Moving along the cycle, we see that LUB is sent to DBR, and finally, DBR is sent back to FRU. One could say that this cycle has "torsion" because one loop around the cycle leaves us in a different orientation from that in which we started. We write this cycle as

$$(\mathsf{UFR\ LUB\ DBR})_+,$$

where the $+$ subscript indicates that the cycle returns to the first piece with its second sticker "facing up." Likewise, the other cycle of corners can be written with a $-$ subscript, as in

$$(\mathsf{URB\ FDR\ ULF\ BDL\ FLD})_-,$$

because the cycle returned to the third sticker of the first piece. A similar ornamentation is given to edge cycles: if such a cycle returns to the other sticker of the first piece, it is given a $+$ subscript. Finally, when a corner or edge cycle returns to the same sticker, no subscript is recorded. Following these definitions and the visual depiction of the cycle decomposition at the bottom of Figure 1, the cycle decomposition of the edges reads:

$$(\mathsf{UF})\ (\mathsf{BL})_+\ (\mathsf{FL\ FR})_+\ (\mathsf{UR\ UL\ BD\ RD\ UB\ RB\ LD\ DF}).$$

The subscript attached to each cycle is what we call the *twist type* of the cycle, and later we prove a constraint on possible collections of twist types. The "oriented" cycle type of a Rubik's Cube position replaces each cycle with its length while preserving the subscript. Our example has the cycle type $(5_-, 3_+)$ for corners, and $(8, 2_+, 1_+, 1)$ for edges.

Given any position, its inverse has the same cycle type, except each corner cycle of twist type $+$ now has twist type $-$, and vice versa. One can think of the twist types of the corners and edges as isomorphic to the integers modulo 3 and 2, respectively.

3. INVARIANTS OF THE RUBIK'S CUBE Disassembling and reassembling a Rubik's Cube may result in a position which is not solvable, and thus not all cycle types are realizable by a solvable position. We briefly review how piece orientations can be defined and how they are used to formulate the solvability criteria. The discovery of these invariants is attributed to Anne Scott (see [**11**, p.13]).

The *cubicle* of a piece is its starting location in the solved state, and a piece can be in different orientations in the same cubicle. When a piece is in its cubicle, it is straightforward to define the orientation of the piece as the amount one needs to rotate the piece in place until it becomes solved. But how do we describe the orientation of a piece away from its cubicle? The standard approach is to define a *frame of reference* by designating, for each corner and edge piece, a *chief sticker*. Then, if piece P is in the cubicle of piece Q, then we can describe the orientation of P based on the relative orientation of their chief stickers. If P is a corner, then we use $+$ (resp. $-$) to indicate that the chief sticker of P is twisted 120 degrees counterclockwise (resp. clockwise) from that of Q. Misoriented edges are denoted by a $+$.

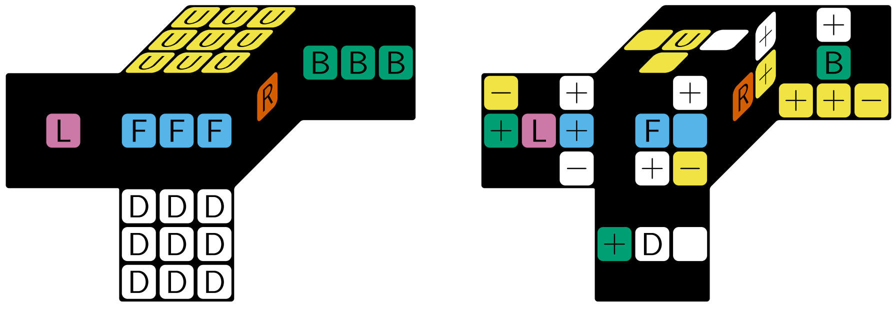


Figure 2. The chief stickers of the canonical frame, and their locations in the example scramble.

The forthcoming results apply for any choice of chief stickers, but later on, it will be helpful to fix one specific frame of reference. The simplest choice is what we call the *canonical frame* shown in Figure 2(a): for each piece, select the U or D sticker, or if the piece does not have such a sticker (in the case of the four "equatorial" edge pieces), the F or B sticker. Figure 2(b) illustrates how the chief stickers determine the orientations of pieces, with respect to this canonical frame, in the example scramble.

Since rotating a piece in place cycles through its different orientations, the group of a specific corner or edge's orientations is isomorphic to the integers modulo 3 or 2, respectively. Not all combinations of corner and edge orientations are viable:

Proposition 3.1 (corner orientation invariant). *In a solvable position, the sum of the corner twists is* $0 \pmod 3$ *with respect to any frame of reference.*

Proposition 3.2 (edge orientation invariant). *In a solvable position, the sum of the edge flips is* $0 \pmod 2$ *with respect to any frame of reference.*

The overloading of the $+$ and $-$ notation for both piece orientations and cycle twist types is not coincidental, though we note that, unlike individual piece orientations, twist types are independent of the frame of reference. Understanding our construction is aided by extending the above two invariants to twist types:

Lemma 3.3. *The sums of twist types of corner and edge cycles are* $0 \pmod 3$ *and* $0 \pmod 2$*, respectively.*

Proof. We may choose a frame of reference so that all but possibly one piece per cycle is correctly oriented. That last piece's orientation matches the twist type of the cycle. ■

In several places, our construction rotates pieces in place, and we note that doing so can only affect twist types, and not the cycle lengths. The above Lemma can save us the effort of calculating those changes, since we can infer the twist type of one cycle by looking at the twist types of all the other cycles. Another related consequence is that solvability can be checked just by looking at the cycle type.

If we ignore the orientations of all the pieces, then a quarter turn of any face applies a 4-cycle on both the corners and edges. Since the solved position consists of two even identity permutations, our final invariant thus reads as follows:

Proposition 3.4 (total permutation invariant). *In a solvable position, the corner and edge permutations have the same parity.*

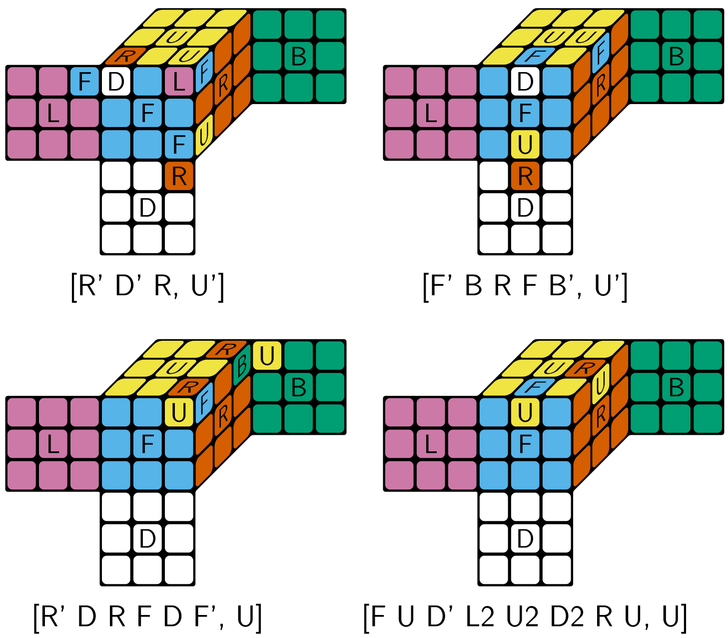


Figure 3. Some well-known commutators that solve the pictured positions.

Any position that violates any one of these invariants is surely unsolvable, but an oft-overlooked step is showing the converse. We first reduce the problem to solving even positions by making any quarter turn on any odd position. Next, we introduce the four commutators in Figure 3 that cycle three corners, cycle three edges, twist two corners, and flip two edges, respectively.

Remark. The first of these commutators belongs to a family of 8-move commutators that permute three corners. Their short lengths make them particularly useful in fewest moves solving, where a cuber works on one scrambled position in a prescribed time limit. A common strategy is to reach a partial solution where only three corners are incorrect. Then, one tries to solve those corners by inserting a commutator in the middle of their partial solution, hoping to cancel out turns in the process.

There is enough wiggle room to permute any three or orient any two pieces using a conjugate of one of these commutators. More concretely, this process involves placing the target pieces in the positions of the affected pieces in Figure 3, applying the commutator, and then carefully undoing those "setup moves." For example, twisting the corners DRF and ULF can be accomplished via the conjugation

R2 F [R' D R F D F', U] F' R2,

since R2 F sends DRF to URB and ULF to RUF. Surprisingly, this is already a complete method for solving the Rubik's Cube: make a quarter turn if the position is odd, place the pieces in their cubicles, and finally, fix their orientations if necessary. If the total permutation invariant is satisfied, then the corner and edge permutations can both be decomposed into products of 3-cycles. The latter two commutators can be used to correct the orientations of all but one corner and one edge. In positions where the corner and edge orientation invariants are satisfied, the remaining two pieces must be correctly oriented, as well. Since conjugates of commutators are themselves commutators ($a[b,c]a^{-1} = [aba^{-1}, aca^{-1}]$), we obtain the easier half of our main result:

Proposition 3.5. *The commutator subgroup of the Rubik's Cube group consists of all even solvable positions.*

Remark. This approach is not too far from actual blindfolded methods for solving the Rubik's Cube. A speedcuber would essentially memorize the cycle decomposition of the inverse position and use a larger set of commutators to permute and orient pieces simultaneously. Fixing the parity in odd positions is actually done at the end of the solution using a sequence that permutes two corners and two edges. One does not know, *a priori*, the parity of the position, and trying to work out how a quarter turn affects the cycle decomposition would be too time-consuming.

4. CONJUGACY IN THE RUBIK'S CUBE GROUP Two group elements g and h are *conjugate* if there is some element x such that $h = xgx^{-1}$. We call x the *conjugator from g to h*. Conjugacy is an equivalence relation, partitioning the group elements into *conjugacy classes*. One approach for representing an arbitrary group element as a commutator is to first factor it into a product gh, where g^{-1} and h are conjugate. If x is a conjugator from g^{-1} to h, then $[g, x]$ is the desired commutator. For this reason, we sometimes refer to the second term of a commutator as a conjugator as well.

In the symmetric group, two permutations are conjugate if and only if they have the same cycle type. A conjugator can be constructed by bijectively mapping each cycle from one permutation to a cycle of the same length in the other permutation. In the alternating group A_n, having the same cycle type is no longer sufficient. For example, the 5-cycles $(1\ 2\ 3\ 4\ 5)$ and $(1\ 2\ 3\ 5\ 4)$ are conjugate in S_5, but all possible conjugators have odd parity. Hence, they belong to different conjugacy classes of A_5.

The cycle-matching procedure can also be applied to Rubik's Cube positions, where the cycles must agree in both length and twist type. Just like in the alternating group, the resulting conjugator might not be solvable. Singmaster [**11**, p.58] describes a pair of positions whose conjugators all violate the total permutation invariant. Here, the role of the symmetric group (i.e., the group to which the conjugators belong) is played by the *constructible group*, the supergroup consisting of all ways of disassembling and reassembling the puzzle without peeling off any stickers. In some cases, we can modify an unsolvable conjugator into a solvable one:

Proposition 4.1. *Suppose that y is a conjugator from x to w, and let z be an element that commutes with x. Then, yz is also a conjugator from x to w.*

Proof. $(yz)x(yz)^{-1} = yzxz^{-1}y^{-1} = y(zz^{-1})xy^{-1} = yxy^{-1} = w.$ ■

Corollary 4.2. *Let $[x, y]$ be a commutator, and let z be an element that commutes with x. Then, $[x, y] = [x, yz]$.*

In contrast to 5-cycles, 3-cycles are always conjugate to one another in A_5: if the conjugator y has odd parity, choose z to be the transposition that swaps the two fixed symbols in x. Now, yz has even parity. In general, if the modifying element z is outside of the subgroup, then y and yz belong to different cosets. Our method makes use of positions belonging to one of two cycle types. In both cycle types, we show that all positions are conjugate to one another by finding appropriate positions z that correct violations of the three solvability invariants.

5. BERTRAM'S THEOREM As one of several methods for constructing commutators in the symmetric group, Bertram [**1**] described when an even permutation can be expressed as a product of two ℓ-cycles. We define the *support* $\mathrm{supp}(P)$ of P to be the set of symbols it moves, and we say that two permutations P and Q *overlap* if some symbol is moved by both permutations, i.e., $\mathrm{supp}(P) \cap \mathrm{supp}(Q) \neq \emptyset$.

Theorem 5.1 (Bertram [1]). *Let P be an even permutation on n symbols, and suppose that P moves $m := |\operatorname{supp}(P)|$ symbols and has c nontrivial cycles. Then, P can be expressed as a product of two overlapping ℓ-cycles, for any $(m+c)/2 \leq \ell \leq n$.*

We can maximize $m + c$ by trying to pack as many transpositions into the permutation as possible, which results in a bound independent of the permutation itself:

Corollary 5.2 (Bertram [1]). *Every even permutation on n symbols can be written as a product of two ℓ-cycles, where $\ell \geq \lfloor 3n/4 \rfloor$.*

Short proofs of Theorem 5.1 can be found in Herzog *et al.* [**5**] and Appendix A. A common feature of these proofs is that the case where $\ell = (m+c)/2$ is solved first, and then additional symbols are incorporated into the cycles. Achieving this second step can be boiled down into two simple statements about multiplying by transpositions:

Fact 1. *Suppose a permutation has two different cycles of the form $(s\ \dots)$ and $(t\ \dots)$. Left- or right-multiplying the permutation by the transposition $(s\ t)$ merges the two cycles together.*

Fact 2. *Given two permutations A and B, right-multiplying A and left-multiplying B by the same order 2 permutation T (such as a transposition) preserves their product, i.e., $AB = (AT)(TB)$.*

Bertram and Herzog [**2**] essentially proved the following result, which can be applied repeatedly to achieve all the remaining cases of Bertram's theorem:

Lemma 5.3. *Let A and B be two overlapping ℓ-cycles in the symmetric group S_n, where $1 < \ell < n$, and let s be a symbol fixed by at least one of A or B. Then, there are two overlapping $(\ell+1)$-cycles A' and B' that both move s satisfying $AB = A'B'$.*

Proof. If s is fixed by both A and B, then let t be a symbol moved by both A and B. Then let $A' = AT$ and $B' = TB$, where T is the transposition $(s\ t)$. Otherwise, assume without loss of generality that s is fixed by A, but not by B. Since A and B are cycles of the same length, by the pigeonhole principle, there must be some symbol u which is fixed by B, but not by A. Just like in the first case, we can multiply A and B by the transposition $(s\ u)$. In both cases, the ℓ-cycles of A and B are being merged with trivial cycles of length 1 to make two $(\ell+1)$-cycles. ■

Figure 4 illustrates both modifications, where the vertices are the symbols and the directed edges are the mappings.

Our method for solving the corners only requires the special case of Bertram's theorem where the cycles permute all of the symbols, i.e., when $\ell = n$. The purpose of describing his result in full detail is to extract out the following technical result used for solving the edges:

Lemma 5.4. *Let $P \in S_{12}$ be an even permutation, and let s be a symbol fixed by P. Then, P can be written as a product of two 10-cycles A and B, where s is moved by both A and B.*

Proof. By Corollary 5.2, P can be expressed as a product of two 9-cycles A and B. Applying Lemma 5.3 on the symbol s results in two 10-cycles. ■

Our application of Bertram's theorem to the Rubik's Cube results in factorizations involving positions we call *pure permutations* and *pure orientations*. A position is said

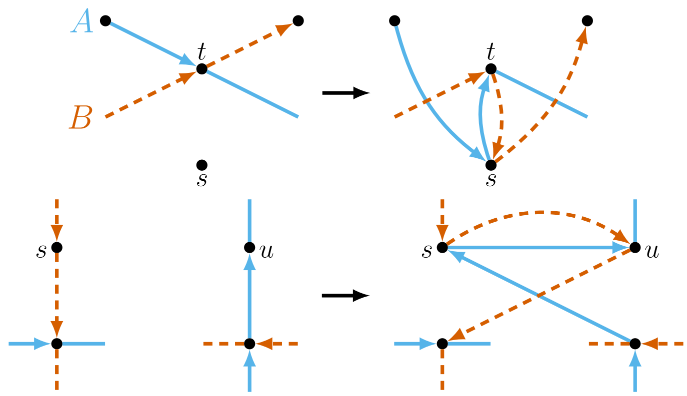


Figure 4. Incorporating new symbols into cycles without affecting the product.

to be a *pure permutation* if every piece is correctly oriented with respect to the canonical frame. If every piece is in its correct cubicle, but in possibly incorrect orientations, then it is a *pure orientation*.

If the orientations of the pieces are ignored, the constructible group collapses into the direct product $S_8 \times S_{12}$. To streamline the notation, the symbol we choose for each piece starts with its chief sticker in the canonical frame, e.g., ULF or BR. For even, solvable positions, each component of its corresponding element in $S_8 \times S_{12}$ is even. Thus, Bertram's theorem can be applied to each component separately. In the reverse direction, every element in $S_8 \times S_{12}$ can be interpreted as a pure permutation.

When we say that we apply Bertram's theorem to a *position* P, we mean we "forget" the piece orientations, apply Bertram's theorem to each piece type, and finally "misremember" the orientations of the pieces to get two pure permutations P_A and P_B. Since this process loses all information on piece orientations, this factorization results in yet another position P_O, a pure orientation satisfying $P = P_A P_B P_O$. Our method for constructing commutators attempts to find a conjugator from P_A^{-1} to $P_B P_O$. These two positions may not have the same cycle type (some cycles may have non-matching twist types), so some pieces may need to be reoriented.

6. CORNERS We solve the corners separately from the edges, and it helps to completely ignore the edges by peeling off all of those stickers. Solvability on the resulting "corners-only" cube is governed by only the corner orientation invariant. The notions of odd and even positions, pure permutations and orientations, and the constructible group carry over into this subgroup.

Consider the "supertwist" position T on the left part of Figure 5. It is a pure orientation, and since all of the eight corners are twisted in the same direction, this position is unsolvable. It also satisfies another key property:

Proposition 6.1. *The supertwist position commutes with every element in the corners-only cube.*

Proof. If P is some arbitrary constructible position, we want to show that $P = TPT^{-1}$ by checking the image of each corner piece. Suppose P maps the corner piece ABC to XYZ. By focusing on a different orientation of the two pieces, an equivalent way of describing this map is to say that BCA is sent to YZX. Thus, TPT^{-1} sends ABC $\mapsto$ BCA $\mapsto$ YZX $\mapsto$ XYZ as well. ■

In other words, the supertwist is in the *center* of the group. The supertwist con-

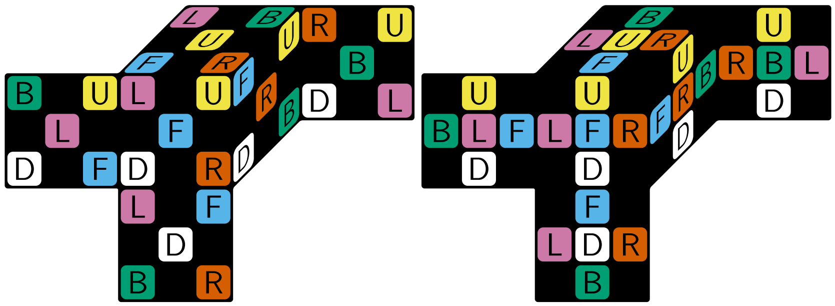


Figure 5. The unsolvable supertwist, and the solvable superflip.

stitutes a universal fix for trying to find a solvable conjugator between corners-only positions:

Lemma 6.2. *Every even, solvable position in the corners-only cube can be solved by a commutator $[C_1, C_2]$, where C_1 and C_2 are solvable, and C_1 is an odd position.*

Proof. Applying Bertram's theorem to an even solvable position C^* results in a factorization $C^* = C_A C_B C_O$, where C_A and C_B are two pure permutations of cycle type (8), and C_O is a pure orientation. Cycle types of (8_+) and (8_-) are impossible due to Lemma 3.3, so both C_A^{-1} and $C_B C_O$ must also have cycle type (8).

Let C_X be any (constructible) conjugator from C_A^{-1} to $C_B C_O$. Then, by Corollary 4.2, C^* can be written as commutators of the form

$$[C_A, C_X] = [C_A, C_X T^k],$$

where k is any integer and T is the supertwist. Since T violates the corner orientation invariant, exactly one of C_X, $C_X T$, and $C_X T^2$ is solvable. ■

In our example scramble, the inverse permutation, after ignoring orientations, has the cycle decomposition

$$(\mathsf{UFR\ DBR\ UBL})(\mathsf{URB\ DFL\ DLB\ ULF\ DRF}),$$

which can be factored into a product of two 8-cycles AB, where

$$A = (\mathsf{UFR\ UBL\ DFL\ URB\ DLB\ DRF\ ULF\ DBR})$$
$$B = (\mathsf{UFR\ UBL\ DBR\ DRF\ ULF\ URB\ DLB\ DFL}).$$

Interpreting these two 8-cycles as positions, the "oriented" cycle decomposition of C_A reads the same as the cycle decomposition of A, but $C_B C_O$ has some corners twisted:

$$C_B C_O = (\mathsf{UFR\ LUB\ BRD\ DRF\ LFU\ BUR\ DLB\ LDF}).$$

The conjugator from C_A^{-1} to $C_B C_O$ where UFR is mapped to itself happens to be unsolvable, but if we compose it with the supertwist, then we obtain the solvable conjugator

$$C_X = (\mathsf{UFR})_+(\mathsf{URB})_-(\mathsf{ULF\ DLB\ FLD\ UBL\ RDB})_-(\mathsf{DRF})_+$$

shown in Figure 6.

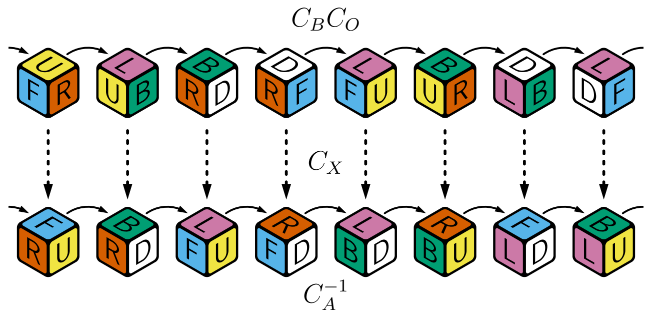


Figure 6. Constructing a corners-only conjugator by matching the two 8-cycles.

7. EDGES Starting with a brand new Rubik's Cube, we now peel off the corner stickers to get the "edges-only" cube. The analogue of the supertwist is known as the *superflip*, shown on the right of Figure 5. The *unsolvability* of the supertwist was crucial in the corners' solution, but unfortunately, the superflip has an even number of misoriented edges, making it solvable. One can use this fact to show that the edges-only positions of cycle type (12) are divided into two conjugacy classes, hence we cannot reuse our earlier method here. To remedy this, we will have to turn to positions of other cycle types. These positions are trickier to work with.

An edge is said to be *flipped in place* if it is in its correct cubicle, but in the wrong orientation. If Z is a subset of the edges, let F_Z denote the pure orientation where an edge is flipped in place if and only if it is in Z. The solvability of F_Z depends on the parity of $|Z|$, and when Z is nonempty, F_Z has order 2. For example, the superflip is the position where Z is the set of all edges. We first describe a simple sufficient condition for conjugacy in the edges-only cube:

Proposition 7.1. *Suppose two positions have the same cycle type and that the cycle type contains a "1", i.e., there is a solved edge. Then, the two positions are conjugate.*

Proof. Suppose that the edge XX is solved in the first position, and consider any conjugator from the first position to the second position. If the conjugator is not solvable, then by Proposition 4.1, multiplying it by F_{XX} (which also violates the edge orientation invariant) yields a solvable conjugator. ∎

The perturbation F_{XX} resembles the one used in our earlier mention of 3-cycles in A_5: it is a transposition of two fixed stickers.

Bertram's theorem can be used to factor any even permutation on 12 symbols into a product of two 10-cycles, but now the positions P_A^{-1} and $P_B P_O$ can have one of three different cycle types. We show that, in all cases, we can match up the cycle types by inserting an appropriate pure orientation F_Z. Our construction splits into two cases depending on the number of edges flipped in place:

Proposition 7.2. *Every even, solvable position in the edges-only cube with at least two edges flipped in place can be expressed as a product of two positions of cycle type* $(10_+, 1_+, 1)$.

Proof. Pick any two edges XX and YY that are flipped in place, and apply Bertram's theorem restricted to the remaining ten edge pieces to obtain a factorization $E_A E_B E_O$, where E_A and E_B are two pure permutations of cycle type $(10, 1, 1)$, and E_O is a pure orientation. Since both E_A and E_B fix XX and YY, E_O must flip those two edges. Thus, $E_B E_O$ has cycle type $(10, 1_+, 1_+)$. Let ZZ be any other edge, and form the pure orientation $F_{\mathsf{XX,ZZ}}$. Since ZZ is part of the 10-cycle of both E_A and E_B, the compositions $E_A F_{\mathsf{XX,ZZ}}$ and $F_{\mathsf{XX,ZZ}} E_B E_O$ are the desired positions of cycle type $(10_+, 1_+, 1)$. In the former, XX becomes flipped in place, and in the latter, YY remains flipped in place. ■

An example of such a position is the superflip. Since all edges are in their cubicles, the product of 10-cycles resulting from Bertram's theorem can be any 10-cycle and its inverse. Tracing through the steps above, with $\mathsf{XX} = \mathsf{UF}$, $\mathsf{YY} = \mathsf{UR}$, and $\mathsf{ZZ} = \mathsf{UB}$, yields the positions

$$E_A F_{\mathsf{UF,UB}} = (\mathsf{UB\ UL\ FR\ FL\ BR\ BL\ DF\ DR\ DB\ DL})_+(\mathsf{UF})_+(\mathsf{UR})$$
$$F_{\mathsf{UF,UB}} E_B E_O = (\mathsf{UB\ DL\ BD\ DR\ FD\ BL\ RB\ FL\ RF\ UL})_+(\mathsf{UR})_+(\mathsf{UF}).$$

Proposition 7.3. *Every even, solvable position in the edges-only cube with at most one edge flipped in place can be expressed as a product of two positions of cycle type* $(10, 1, 1)$.

Proof. By Lemma 5.4, the inverse position has the factorization $E_A E_B E_O$, where E_A and E_B are pure permutations of cycle type $(10, 1, 1)$, and the edge flipped in place (if present) is part of the 10-cycle in both E_A and E_B.

In the position $E_B E_O$, there may be edges which are flipped in place. Any such edge XX must be part of the 10-cycle of E_A, since otherwise, that edge would be flipped in place in the original position, unmoved by both cycles. Pick any edge ZZ belonging to both 10-cycles. We insert two copies of $F_{\mathsf{XX,ZZ}}$ in between E_A and $E_B E_O$. In E_A, both of those edges belong to the 10-cycle, so $E_A F_{\mathsf{XX,ZZ}}$ has the same cycle type $(10, 1, 1)$. In $F_{\mathsf{XX,ZZ}} E_B E_O$, the twist types of the 10-cycle and the 1-cycle corresponding to XX flip. If $E_B E_O$ has another edge YY flipped in place, we can repeat the same procedure, multiplying both terms by $F_{\mathsf{YY,ZZ}}$. After these adjustments, both $E_A F_Z$ and $F_Z E_B E_O$, for some appropriate set of edges Z, have cycle type $(10, 1, 1)$. ■

In our running example, there is one flipped edge BL. Applying Bertram's theorem to the inverse position, making sure that both 10-cycles move BL, produces the pair of permutations

$$A = (\mathsf{UF\ FL\ FR\ BL\ BR\ DF\ UB\ UL\ DR\ DB})(\mathsf{UR})(\mathsf{DL})$$
$$B = (\mathsf{UF\ UL\ DR\ UR\ DF\ UB\ DL\ BR\ BL\ FL})(\mathsf{DB})(\mathsf{FR})$$

One can check that in the resulting position $E_B E_O$, the edge FR is flipped in place. We can fix this by multiplying both positions by $F_{\mathsf{FR,UB}}$. The resulting cycle decompositions are

$$E_A F_{\mathsf{FR,UB}} = (\mathsf{UF\ FL\ RF\ LB\ RB\ FD\ UB\ UL\ DR\ DB})(\mathsf{UR})(\mathsf{DL})$$
$$F_{\mathsf{FR,UB}} E_B E_O = (\mathsf{UF\ LU\ DR\ UR\ DF\ BU\ LD\ RB\ BL\ FL})(\mathsf{DB})(\mathsf{FR})$$

The two previous constructions cover all possible cases, and since both cycle types contain a solved edge, Proposition 7.1 shows that both factorizations can be expressed as commutators with solvable conjugators:

Lemma 7.4. *Every even, solvable position in the edges-only cube can be solved by a commutator* $[E_1, E_2]$*, where* E_1 *is an odd position.*

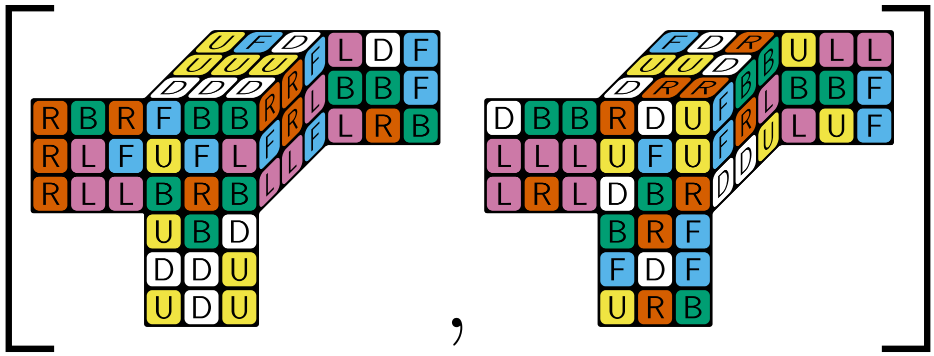


Figure 7. A commutator solving the example scramble.

One possible conjugator from $(E_A F_{\mathsf{FR,UB}})^{-1}$ to $(F_{\mathsf{FR,UB}} E_B E_O)$ is

$$E_X = (\mathsf{UR\ RD\ DB})(\mathsf{UF\ LF\ RF\ LD\ DF\ LU})_+(\mathsf{BR})_+(\mathsf{BL})(\mathsf{UB}).$$

8. TOTAL PERMUTATION Solvable positions in the corners-only and edges-only cubes can be combined into a solvable position of the original Rubik's Cube if and only if the total permutation invariant is satisfied. The choice to factor both the corners and edges as a product of two odd permutations serves two purposes: the first term of the combined commutator automatically satisfies the invariant, and the second term can be easily corrected if it does not:

Theorem 8.1. *Every even position of the Rubik's Cube can be solved by a commutator.*

Proof. From Lemmas 6.2 and 7.4, the corners and edges can be solved separately by two commutators $[C_1, C_2]$ and $[E_1, E_2]$, where both C_1 and E_1 are odd positions. If C_2 and E_2 differ in parity, then by Corollary 4.2, we may replace E_2 with $E_2 E_1$, since E_1 is odd and trivially commutes with itself. Now, C_2 and E_2 have the same parity and can be combined into two positions that satisfy all three invariants. ■

The conjugators we found earlier, C_X and E_X, have different parities. We replace E_X with

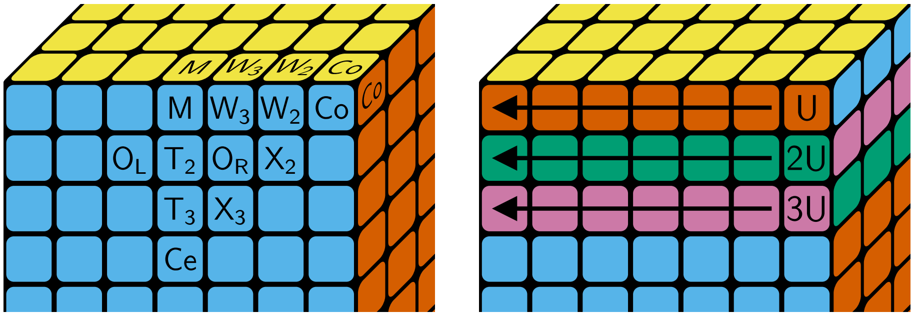
$$E_X E_A = (\mathsf{UF\ FR\ DL\ UB\ UL\ FL\ BL\ BR\ FD\ DR})_+(\mathsf{UR\ BD})_+,$$

and now both conjugators are even. Let P_A be the combination of C_A and E_A, and P_X be the combination of C_X and $E_X E_A$. The final commutator $[P_A, P_X]$ is shown in Figure 7, where P_A and P_X can be generated by the following sequences:

$$P_A = \mathsf{B2\ D\ F2\ U\ R2\ U\ R'\ U\ L\ U'\ F\ U\ F'\ R'\ F2\ U\ L}$$

$$P_X = \mathsf{L\ B'\ D'\ B2\ L'\ B2\ F'\ L\ D'\ F\ D2\ R\ B'\ R'\ F2\ L\ F\ U'}$$

9. THE $n \times n \times n$ RUBIK'S CUBE Higher-order cubes introduce new piece types that generalize those of the original $3 \times 3 \times 3$ Rubik's Cube. Figure 8(a) depicts the different piece types of the $7 \times 7 \times 7$ cube (with nonstandard names). What we have labeled Ce, M, and Co in the figure can be thought of as a $3 \times 3 \times 3$ cube embedded in the



(a) (b)

Figure 8. Piece types (a) and the three classes of moves (b) in the $7{\times}7{\times}7$ cube.

bigger cube, so our earlier method can be used to solve these piece types. The remaining pieces actually have a much simpler solution due to the lack of piece orientations—even the generalized edge piece types W_2 and W_3, which have two stickers per piece, cannot be flipped in place. Thus, each new piece type looks like a symmetric group S_{24}, subject to some new total permutation constraints.

Figure 8(b) shows the different kinds of turns with the U axis as an example. X and mX turns, for $m \geq 2$, $\mathsf{X} \in \{\mathsf{U}, \mathsf{F}, \mathsf{R}, \mathsf{B}, \mathsf{L}, \mathsf{D}\}$, are often referred to as *face turns* and *slice turns*, respectively. Each set of turns at a particular "depth" induces its own total permutation invariant. For example, face turns on the $7{\times}7{\times}7$ apply a 4-cycle to each generalized center piece (besides Ce), M, and Co, but apply two 4-cycles to each generalized edge piece (besides M).

Consider any position that can be reached from the solved state using an even number of mX slice turns, for each m. In such positions, the parity of any generalized center piece (besides Ce) matches that of the embedded $3{\times}3{\times}3$, and the parity of any generalized edge piece (besides M, when n is odd) is always even. After finding a solution to the embedded $3{\times}3{\times}3$, we may solve all the remaining piece types one by one, accomodating any total permutation requirements with a unified approach:

Lemma 9.1. *Every even permutation of S_{24} can be expressed as a commutator $[X, Y]$, for any choice of parities for X and Y.*

Proof. By Bertram's theorem, every even permutation of S_{24} can be written as a product AB, where A and B are either two 21-cycles, or two 22-cycles. Then, $AB = [A, Y]$, where Y is any conjugator from A^{-1} to B. If Y does not have the desired parity, then consider any two symbols a_1 and a_2 fixed by A. By Corollary 4.2, right-multiplying Y with the transposition $(a_1\ a_2)$ yields a conjugator of the opposite parity. ■

There are some minor technicalities when working with larger order cubes regarding the ambiguity of the solved state: center stickers in the usual $n{\times}n{\times}n$ cube are indistinguishable, and when n is even, the standard set of moves can be combined to rotate the entire cube (e.g. U 2U 2D' D' on the $4{\times}4{\times}4$). Fixing one of those solved states as the identity permutation resolves these issues, and hence the positions of the $n{\times}n{\times}n$ cube describe a group. The commutator subgroup once again consists of the positions where the parity of each piece type is even.

Theorem 9.2. *Every element of the commutator subgroup of the $n{\times}n{\times}n$ cube group can be solved by a commutator.*

10. CONCLUSION Our method produces a single commutator for each even position of the Rubik's Cube, but a few questions remain about the variety of such commutators. For the $3\times3\times3$ cube, our commutators correspond to one of two conjugacy classes. With a little extra work, Proposition 7.3 can be extended to show that only cycle type $(10_+, 1_+, 1)$ is needed. Which other cycle types are sufficient by themselves? Cycle type (12), the one we had to avoid earlier, is particularly tantalizing: it would be more aesthetically pleasing to have a solution that mirrors that of the corners. Perhaps one can guarantee conjugacy using more deliberate choices of 12-cycles, rather than ones arbitrarily generated by Bertram's theorem.

The commutators in Figure 3 have intuitive explanations and use only a few moves, but on the other hand, our commutators are complicated and often long. Rokicki, Kociemba, Davidson, and Dethridge [**10**] showed that the diameter of the Rubik's Cube group is 20 [**10**], so each of our commutators uses at most 80 turns. Is there any way to guarantee fewer turns? Answering this will almost certainly require computational techniques, since positions belonging to the same conjugacy class can have different distances to the solved state.

The commutator problem for other twisty puzzles is open. The *Megaminx* puzzle is the dodecahedral analogue of the Rubik's Cube, and since each face is pentagonal, every solvable position has an even permutation applied to its corner and edges. Thus, the Megaminx group can be shown to be equal to its commutator subgroup. We expect that any solution to the commutator problem for this puzzle will be significantly more complicated than our solution for cubic puzzles due to this parity restriction.

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A. ANOTHER PROOF OF BERTRAM'S THEOREM

Proof of Theorem 5.1. By Lemma 5.3, it suffices to show that P can be expressed as a product of overlapping ℓ_0-cycles, where $\ell_0 := (m+c)/2$.

We start by writing P as a product of two conjugate permutations by handling cycles one or two at a time, decomposing them as products of cycles on the same

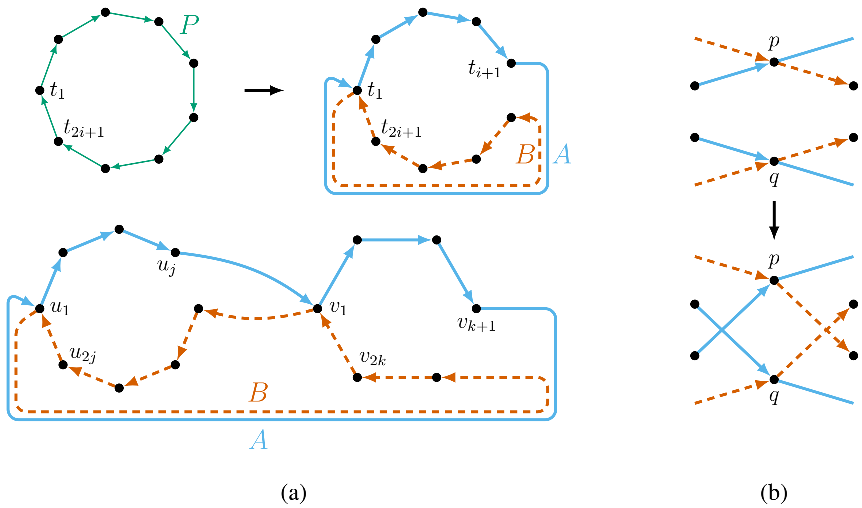

Figure 9. Splitting one odd cycle and two even cycles into products of two cycles (a), and joining pairs of disjoint cycles (b).

symbols. This method was first seen in Ore [**9**], but our choices of cycles are slightly more unified. Following Figure 9(a), each nontrivial odd cycle $(t_1\ t_2\ \dots\ t_{2i+1})$ in P can be written as a product of two $(i+1)$-cycles

$$(t_1\ t_2\ \dots\ t_{i+1})(t_1\ t_{i+2}\ t_{i+3} \dots\ t_{2i+1}),$$

and each pair of even cycles $(u_1\ u_2\ \dots u_{2j})(v_1\ v_2\ \dots v_{2k})$ can be written as a product of two $(j+k+1)$-cycles

$$(u_1\ u_2\ \dots\ u_j\ v_1\ v_2\ \dots\ v_{k+1})(u_1\ v_{k+2}\ v_{k+3} \dots\ v_{2k}\ v_1\ u_{j+1}\ u_{j+2}\ \dots\ u_{2j}).$$

Note that the latter construction works even in the smallest cases where either cycle is a transposition. The first and second terms of each product combine to form permutations A and B, respectively.

In each nontrivial cycle, each symbol is moved by exactly one of A or B, except the "first" symbol (i.e., t_1, u_1, v_1 in our earlier notation), which is moved by both. Thus, the combined number of moved symbols is $|\operatorname{supp}(A)| + |\operatorname{supp}(B)| = m + c$, where $m = |\operatorname{supp}(P)|$ and c is the number of nontrivial cycles. Since A and B are of the same cycle type, they must each move exactly $\ell_0 = (m+c)/2$ symbols.

To string all of the cycles together, we use the fact that each pair of cycles that were used to form A and B overlap at some symbol. Consider cycles of the form $(p\ \dots)$ and $(q\ \dots)$ in both A and B. Multiplying by the transposition $T = (p\ q)$ merges those two cycles of both A and B, as seen in Figure 9(b). The compositions AT and TB have the same supports as A and B, respectively. The two new cycles still overlap at symbols p and q, so this process can be repeated until all cycles are merged into one long cycle. Since the supports stayed the same throughout this procedure, we conclude that the final permutations are overlapping ℓ_0-cycles.

■