Face distributions of embeddings of complete graphs

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Abstract

A longstanding open question of Archdeacon and Craft asks whether every complete graph has a minimum genus embedding with at most one nontriangular face. We exhibit such an embedding for each complete graph except for K_8, and we go on to prove that no such embedding can exist for this graph. Our approach also solves a more general problem, giving a complete characterization of the possible face distributions (i.e. the numbers of faces of each length) realizable by minimum genus embeddings of each complete graph. We also tackle analogous questions for nonorientable and maximum genus embeddings.

1 Introduction

The celebrated Map Color Theorem of Ringel, Youngs, *et al.* [18] boils down to the fact that for $n \geq 3$, K_n, the complete graph on n vertices, can be embedded in a sphere with

$$I(n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

handles. Equivalently, K_n can be embedded in the orientable surface of genus $I(n)$. These embeddings are provably minimal in terms of genus, as they match a lower bound given by the Euler polyhedral equation. Starting with the work of Lawrencenko *et al.* [14], one direction of continued research on this topic examines the number of nonoisormophic *minimum genus* embeddings, and several different approaches (see, e.g., Bonnington *et al.* [2], Korzhik and Voss [12], Goddyn *et al.* [5]) have yielded families of embeddings whose sizes are exponential in the number of vertices.

The main combinatorial technique for finding minimum genus embeddings of complete graphs are current graph constructions, where an embedding of a smaller, edge-labeled graph can be used to generate a highly symmetric embedding of a much larger graph. The proof generally proceeds in two steps:

- A *regular* step which involves finding a suitable current graph for triangularly embedding a graph that is close to complete (e.g. an embedding of a complete graph minus three edges).
- An *additional adjacency* step which modifies the embedded graph so it becomes complete (e.g. using a handle to add the three missing edges).

When $n \equiv 0, 3, 4, 7 \pmod{12}$, the embedding is a triangulation of the surface and no additional adjacency step is necessary. For the other cases, Korzhik and Voss [13] exhibit exponentially many embeddings by modifying the current graphs used in the regular step. Their proof that the different embeddings are nonisomorphic involves showing that a bijective mapping between the nontriangular faces cannot be extended to the whole embedding.

We exhibit new embeddings of complete graphs that are nonisomorphic for a more fundamental reason: the distributions of the face lengths are different. Archdeacon and Craft [1] ask whether or not every complete graph has a minimum genus embedding that is *nearly triangular*, one where at most one face is nontriangular. We show that K_8 is the only complete graph without such embedding. In the process, we have found simpler families of current graphs for $n \equiv 1, 8 \pmod{12}$. We note that for most of the complete graphs, the original constructions did not produce nearly triangular embeddings (see the exposition in Korzhik and Voss [13]).

One can also ask if there are minimum genus embeddings which manifest all other possible combinations of nontriangular faces (e.g. two quadrangular faces), as permitted by the Euler polyhedral equation. Besides the aforementioned K_8, it turns out that the only other complete graph which does not realize all its predicted embedding types is K_5. The results in this paper can been seen as a step in understanding the embedding polynomials (as introduced by Gross and Furst [7]) of the complete graphs—we fully determine which coefficients corresponding to minimum genus embeddings are nonzero.

In Sections 2 and 3, we review some background on topological graph theory and current graphs. We prove the main result in Section 5, where the different cases are handled in roughly increasing difficulty of the additional adjacency problem. Some variations of the original problem are solved in Sections 6 and 7, and some potential future directions are outlined in Section 8.

2 Background on graph embeddings

We assume familiarity with topological graph theory, especially topics covered in Gross and Tucker [8]. A complete proof of the Map Color Theorem can be found in Ringel [17].

An *embedding* of a graph G in a surface S is an injective map $\phi : G \to S$. We restrict our attention to *cellular* embeddings, those where $S \setminus \phi(G)$ decomposes into a disjoint union of open disks called *faces*, denoted by $F(G, \phi)$. With the exception of Sections 6 and 7, we consider only embeddings in orientable surfaces. A k-sided face can be described by a cyclic sequence $[v_1, v_2, \ldots, v_k]$ of the vertices that appear along the boundary of the face. We sometimes call a 3-sided face a *triangle*, a 4-sided face a *quadrilateral*, and so on. When dealing with embeddings in orientable surfaces, we choose an orientation and apply it consistently to all the faces and the descriptions of their boundaries. As a result, a face described by the cyclic sequence $[a, b, c]$ cannot be described by the reverse sequence $[a, c, b]$.

Let $V(G)$ and $E(G)$ denote the vertex and edge sets of G, respectively, and let S_g denote the surface of genus g, i.e., a sphere with g handles. The fundamental relation governing cellular embeddings is the *Euler polyhedral equation*, which states that for an embedding $\phi : G \to S_g$, we have

$$|V(G)| - |E(G)| + |F(G, \phi)| = 2 - 2g.$$

The *minimum genus* $\gamma(G)$ of a graph G is the smallest value g such that G has an embedding in S_g. By the Euler polyhedral equation, such an embedding has the most faces out of any embedding of G.

An embedding of a graph is said to be *triangular* if all its faces are triangular, and *nearly triangular* if at most one face is not triangular. For a simple graph on three or more vertices, a triangular embedding maximizes the number of faces, so it is impossible for there to be an embedding in a surface of smaller genus. The starting point of the Map Color Theorem and many other graph embedding problems is the following refinement of the Euler polyhedral equation:

Proposition 2.1. *If a graph G has a triangular embedding in S_g, then number of edges in G is*

$$|E(G)| = 3|V(G)| - 6 + 6g.$$

From this relationship, we can figure out what types of minimum genus embeddings are permissible under the Euler polyhedral equation. Some complete graphs, K_7 for example, do have a triangular embedding in some surface, but others do not. However, we can triangulate the nontriangular faces with additional edges without increasing the genus. Substituting "K_n plus t edges" into Proposition 2.1 yields

$$6n + 12g - 12 = 2(|E(K_n)| + t) = n(n-1) + 2t.$$

To remove the genus parameter g, we take the resulting equation modulo 12:

$$2t \equiv -(n-3)(n-4) \pmod{12}.$$

Like in the Map Color Theorem, the analysis now breaks down into twelve Cases (with a capital "C") depending on the residue n mod 12. Define $t(n)$ to be the smallest nonnegative integer t satisfying the above congruence. Initially, we observe the following:

- If $n \equiv 0, 3, 4, 7 \pmod{12}$, $t(n) = 0$.
- If $n \equiv 2, 5 \pmod{12}$, $t(n) = 5$.
- If $n \equiv 1, 6, 9, 10 \pmod{12}$, $t(n) = 3$.
- If $n \equiv 8, 11 \pmod{12}$, $t(n) = 2$.

One way of rephrasing the genus formula for the complete graphs is to say that there exist triangular embeddings of "K_n plus $t(n)$ edges" for all $n \geq 3$. The Euler polyhedral formula guarantees that such an embedding is necessarily of minimum genus.

The *face distribution* (or *region distribution* [22]) of an embedding is the sequence $f_1, f_2, \ldots$ where f_i is the number of faces of length i. For example, a minimum genus embedding of K_7 triangulates the torus, so its face distribution is

$$0, 0, 14, 0, 0, \ldots$$

The primary goal of the present paper is to classify the different possible face distributions for minimum orientable genus embeddings of complete graphs. For the residue classes $n \not\equiv 0, 3, 4, 7 \pmod{12}$, we can enumerate all possible face distributions permitted by the Euler polyhedral equation by partitioning the $t(n)$ "chordal" edges into different faces. For example, for $n = 14$, we have $t(14) = 5$. As seen in Figure 1, distributing all five additional edges into the same face gives us an 8-sided face, but we could distribute the edges in a different way to get one 6-sided face and one 5-sided face.

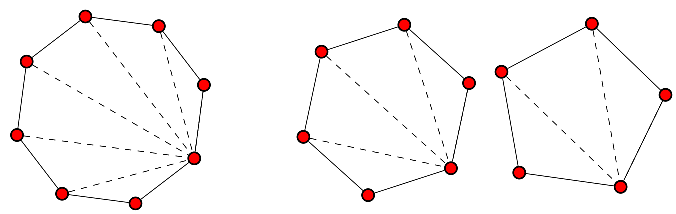

Figure 1: The missing chords could be partitioned among faces in a few different ways.

Instead of writing out face distributions in full and counting all the triangular faces, we say that an embedding is of *type* $(a_1, \ldots, a_i)$, if it has faces of length $a_1, a_2, \ldots, a_i$, where $a_1 \geq a_2 \geq \cdots \geq a_i > 3$, and all the other faces are triangular. In this terminology, K_{14} could have embeddings of type (8) and (6, 5). In general, if $t(n) = b_1 + \cdots + b_j$ is a partition of $t(n)$ into positive integers b_i, we are looking for an embedding of type $(b_1 + 3, b_2 + 3, \ldots, b_j + 3)$. Thus, we need to find the following:

- For $n \equiv 2, 5 \pmod{12}$, types (8), (7, 4), (6, 5), (6, 4, 4), (5, 5, 4), (5, 4, 4, 4), and (4, 4, 4, 4, 4).
- For $n \equiv 1, 6, 9, 10 \pmod{12}$, types (6), (5, 4), (4, 4, 4).
- For $n \equiv 8, 11 \pmod{12}$, types (5) and (4, 4).

Our main result states that all these predicted embedding types exist, except for two complete graph:

Theorem 2.2. *For all $n \geq 3$, $n \neq 5, 8$, and for every partition*

$$t(n) = b_1 + b_2 + \cdots + b_j,$$

of $t(n)$ into positive integers $b_1 \geq b_2 \geq \cdots \geq b_j$, there exists an embedding of type

$$(b_1 + 3, b_2 + 3, \ldots, b_j + 3)$$

of the complete graph K_n. The graph K_5 has minimum genus embeddings only of type (8), (7, 4), (6, 4, 4), (5, 5, 4), and (4, 4, 4, 4, 4), and K_8 has minimum genus embeddings only of type (4, 4).

A special case of this result answers the original question of Archdeacon and Craft [1], showing that:

Corollary 2.3. *For $n \geq 3$, $n \neq 8$, there exists a nearly triangular minimum genus embedding of K_n.*

We first describe the exceptional cases to Theorem 2.2, namely $n = 5, 8$. A direct enumeration over all rotation systems of K_5 reveals that some predicted embedding types in the torus do not exist:

Proposition 2.4 (see Gagarin *et al.* [4] or White [22, p.270])**.** *K_5 has embeddings of type (8), $(7,4)$, $(6,4,4)$, $(5,5,4)$, and $(4,4,4,4,4)$, but no embeddings of type $(6,5)$ or $(5,4,4,4)$.*

For $n = 8$, the nearly triangular case is ruled out by another nonexistence result for triangular embeddings, which was also verified indepedently through exhaustive search. Thus, any minimum genus embedding of K_8 must have type $(4,4)$. To prove this we need a straightforward fact about 5-sided and 6-sided faces. We say a face is *simple* if it is not incident with the same vertex more than once.

Lemma 2.5. *Let G be a simple graph with minimum degree 2. For any orientable embedding of G, all 5-sided faces are simple. A 6-sided face can have at most one repeated vertex—in particular, if there is a repeated vertex x, the face is of the form $[a,b,x,c,d,x]$, where a,b,c,d,x are all distinct.*

Proof. Suppose some vertex v appears twice in some 5-sided face. The face cannot be of the form $[\dots v, v \dots]$, otherwise there would be a self-loop at v. On the other hand, the face also cannot be of the form $[\dots v, w, v \dots]$ for some vertex w, because otherwise w would have degree 1, or there would be more than one edge incident with both v and w.

By the same reasoning, if a vertex x appears twice on the boundary of a 6-sided face, those two instances must appear "opposite" each other, i.e., the face is of the form $[a,b,x,c,d,x]$. Suppose two vertices a and b each appear twice on the same face. Without loss of generality, the face must be of the form $[a,b,c,a,b,d]$. However, since the edge (a,b) is traversed twice in the same direction, this would imply that the embedding is in a nonorientable surface. □

Proposition 2.6. *K_8 does not have a nearly triangular minimum genus embedding.*

Proof. If such an embedding existed, it would be in the surface S_2 and have a simple 5-sided face by Lemma 2.5 and the fact that $t(8) = 5$. By subdividing the face with a new vertex, we obtain a triangular embedding of the graph $K_9 - K_{1,3}$ in S_2. However, Huneke [9] showed that no such embedding exists. □

For the Cases where $t(n) = 2$ or 3, it turns out that practically all of the difficulty is in finding the nearly triangular embedding, i.e. the embeddings of types (5) and (6), respectively. Using those embeddings, it is straightforward to obtain the other types.

Proposition 2.7. *If K_n has an orientable embedding of type (5) (resp. type (6)), then it has an embedding of type $(4,4)$) (resp. types $(5,4)$ and $(4,4,4)$).*

Proof. In the embedding of type (5), the 5-sided face f is simple by Lemma 2.5, so if f is of the form $[\dots a, b, c \dots]$, a must be different from c, and the edge (a,c) is not incident with this face. If we delete the edge (a,c) and add it back in as a chord of f, we get an embedding of type $(4,4)$.

Applying Lemma 2.5 again, suppose the 6-sided face in an embedding of type (6) is of the form $[a, v, w, a', x, y]$, where a and a' are possibly the same vertex. Like in the previous case, we alter the positions of edges (v, w) and (x, y) so that they become chords. Figure 2 illustrates this procedure, where the dashed and thickened lines represent the old and new locations of the edges, respectively. The result is an embedding of type $(4, 4, 4)$. Applying this procedure to just one of the edges yields an embedding of type $(5, 4)$. □

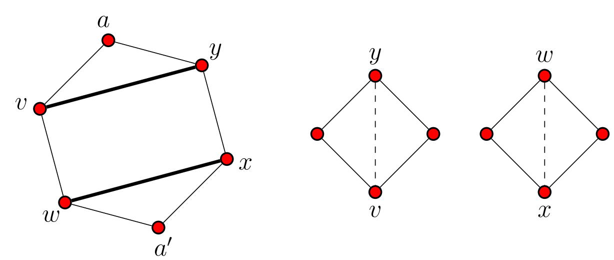

Figure 2: Changing an embedding of type (6) into one of type $(4, 4, 4)$.

Changing the location of an existing edge to a nontriangular face is prevalent in this paper. We call such an operation a *chord exchange* or say that we are *exchanging the chord* (u, v). When multiple chords need to be exchanged in a specific order, we express the sequence of chord exchanges as

$$(u_1, v_1) \to (u_2, v_2) \to \cdots \to (u_i, v_i).$$

3 Rotation systems and current graphs

Let G be a graph, possibly with parallel edges or self-loops. If we give an arbitrary orientation to each of the edges in $E(G)$, then associated with each edge e are two arcs e^+ and e^- that are in opposite directions. We denote the set of such arcs by $E(G)^+$. A *rotation* at vertex $v \in V(G)$ is a cyclic permutation of the arcs in $E(G)^+$ leaving v. A *rotation system* Φ of G is an assignment of a rotation to each vertex of G. In the case of a simple graph, we only need to specify a cyclic ordering of the neighbors of v, thus rotation systems of simple graphs are often written as a table of vertices, where each row corresponds to a rotation at some vertex. For an embedding in an oriented surface, we can obtain a rotation system by considering the clockwise order of edges incident with each vertex, and a rotation system can be converted into a cellular embedding via the face-tracing algorithm.

Our main tools for constructing triangular embeddings of large dense graphs are known as *current graphs*. We describe them slightly informally here—a rigorous topological treatment can be found in Gross and Tucker [8]. For the purposes of this paper, a *current graph* (G, ϕ, α) consists of a graph G, an embedding $\phi : G \to S$ in an orientable surface, and an arc-labeling $\alpha : E(G)^+ \to \mathbb{Z}_n$ with elements from a cyclic group $\mathbb{Z}_n$ such that for all pairs of opposing arcs e^+ and e^-, $\alpha(e^+) = -\alpha(e^-)$. The group $\mathbb{Z}_n$ is referred to as the *current group* and the arc labels are referred to as *currents*. Except for two current graphs described in Appendix A, we only consider *index 1* current graphs, ones where there is one face in the embedding ϕ. The boundary of this face is called the *circuit*, and it consists of a cyclic sequence of arcs $(a_1, a_2, \ldots, a_i)$. The *log* of a circuit $(\alpha(a_1), \alpha(a_2), \ldots, \alpha(a_i))$ replaces each of the arcs with their associated currents.

An example of a current graph is given in Figure 3, where solid and hollow vertices indicate clockwise and counterclockwise rotations, respectively. Let the *excess* of a vertex v be the sum of the currents on the arcs entering v. All the vertices of degree 3 have zero excess, which we refer to as satisfying *Kirchhoff's current law (KCL)*, while all the vertices of degree 1 have nonzero excess.

We take note of the different excesses on the vertices of degree one: the one on the right of Figure 3 produces two consecutive instances of the order 2 element of $\mathbb{Z}_{18}$ in the log of the circuit. This repetition indicates a doubled perfect matching in the final embedding, which can be eliminated by removing one instance of that element from the log. We follow the convention of omitting the degree 1 vertex as a reminder of this convention. The other two vertices of degree 1 are labeled with letters and represent one of each of the following types of *vortices*:

(T1) A vertex of degree 1 whose excess generates the entire current group.

(T2) A vertex of degree 1 whose excess generates the index 2 subgroup of the current group.

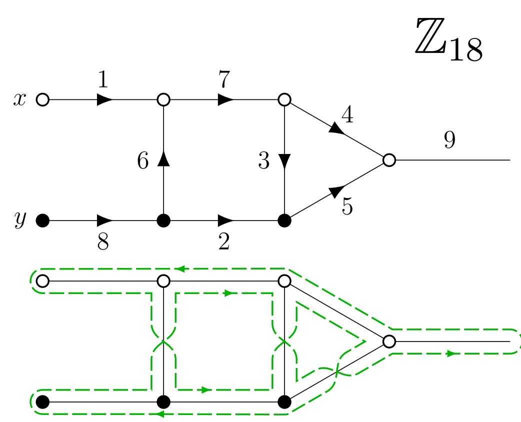

Figure 3: A current graph and its circuit.

That is, the vertices labeled x and y are vortices of type (T1) and (T2), respectively. These letters mark the presence of large nontriangular faces, and we augment the log of the circuit by adding those letters as we encounter them. The final log of the circuit is thus:

$$17 \quad x \quad 1 \quad 12 \quad 2 \quad 15 \quad 4 \quad 13 \quad 16 \quad 10 \quad y \quad 8 \quad 6 \quad 7 \quad 3 \quad 5 \quad 9 \quad 14 \quad 11.$$

To construct the derived embedding of this current graph, we first ignore the letters in the log, and for each element $k \in \mathbb{Z}_{18}$ of the current group, we add k to each numbered element of the log to obtain the rotation at vertex k. The degree 3 vertices induce triangular faces due to KCL, and as mentioned earlier, the vortices of type (T1) and (T2) induce large nontriangular faces. If a vortex of type (T1) has excess γ, the face is a Hamiltonian cycle of the form

$$[0, -\gamma, -2\gamma, -3\gamma, \ldots, 3\gamma, 2\gamma, \gamma],$$

and if a vortex of type (T2) has excess δ, there are two faces, each incident with half of the numbered vertices, of the form

$$[0, -\delta, -2\delta, \ldots, 2\delta, \delta] \text{ and } [1, 1-\delta, 1-2\delta, \ldots, 1+2\delta, 1+\delta].$$

After constructing this derived embedding, we subdivide each of these faces with a vertex and connect that vertex to all the vertices along the face. In our embedding, we have $\gamma = -1$ and $\delta = -8$ for the vortices labeled x and y, respectively, so we add new vertices x, y_0, and y_1 whose induced rotations are

$$\begin{array}{rrrrcrr}
x. & 0 & 17 & 16 & \ldots & 2 & 1 \\
y_0. & 0 & 10 & 2 & \ldots & 16 & 8 \\
y_1. & 1 & 11 & 3 & \ldots & 17 & 9
\end{array}.$$

Subsequently, the rotations at the 18 numbered vertices become

$$\begin{array}{rrcrrrrrrrrcrrrrrrrr}
0. & 17 & x & 1 & 12 & 2 & 15 & 4 & 13 & 16 & 10 & y_0 & 8 & 6 & 7 & 3 & 5 & 9 & 14 & 11 \\
1. & 0 & x & 2 & 13 & 3 & 16 & 5 & 14 & 17 & 11 & y_1 & 9 & 7 & 8 & 4 & 6 & 10 & 15 & 12 \\
2. & 1 & x & 3 & 14 & 4 & 17 & 6 & 15 & 0 & 12 & y_0 & 10 & 8 & 9 & 5 & 7 & 11 & 16 & 13 \\
\vdots & & & & & & & & & & & & & & & & & & & \\
16. & 15 & x & 17 & 10 & 0 & 13 & 2 & 11 & 14 & 8 & y_0 & 6 & 4 & 5 & 1 & 3 & 7 & 12 & 9 \\
17. & 16 & x & 0 & 11 & 1 & 14 & 3 & 12 & 15 & 9 & y_1 & 7 & 5 & 6 & 2 & 4 & 8 & 13 & 10
\end{array}$$

One can check that the entire embedding is triangular, and that all the numbered vertices are adjacent to one another. The current graph in Figure 3, and all others considered in this paper, satisfy the following "construction principles":

(C1) Each vertex of G has degree 3 or 1.

(C2) Each nonzero element of the current group $\mathbb{Z}_n$ appears in the log of the circuit exactly once.

(C3) ϕ is a one-face embedding, i.e., the current graph is of index 1.

(C4) For each vertex of degree 3, KCL is satisfied.

(C5) If the current group has even order, then the order 2 element must be assigned to an arc incident with a vertex of degree 1.

(C6) Each vortex is of type (T1) or (T2).

The familes of current graphs we will encounter contain ladder-like subgraphs, like on the left of Figure 4. The additional adjacency steps use only a small part of a current graph, so we simplify our drawings by using labeled boxes to designate ladder-like subgraphs.

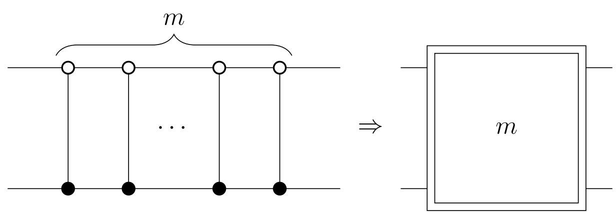

Figure 4: For brevity, ladders are replaced with boxes indicating the number of "rungs."

4 Connecting three vertices with a handle

We make use of the following operation on the rotation at a vertex, which is a variation of Ringel's construction (e.g., Figure 5.2 in [17]) for merging three faces incident with the same vertex:

Modification 4.1. Let ϕ be a triangular embedding, and let v be a vertex with neighbors x, y, z. Suppose the rotation at v in ϕ is of the form

$$v. \quad x \quad a_1 \dots a_i \quad y \quad b_1 \dots b_j \quad z \quad c_1 \dots c_k.$$

Delete the edges (v, x), (v, y), and (v, z), and rewrite the rotation at v as

$$v. \quad a_1 \dots a_i \quad c_1 \dots c_k \quad b_1 \dots b_j.$$

Applying Modification 4.1 on a triangular embedding of genus g yields an embedding of genus $g + 1$ with the 12-sided face

$$[a_1, x, c_k, v, b_1, y, a_i, v, c_1, z, b_j, v],$$

as illustrated in Figure 5. In particular, the number of edges and faces decreased by 3 and 5, respectively.

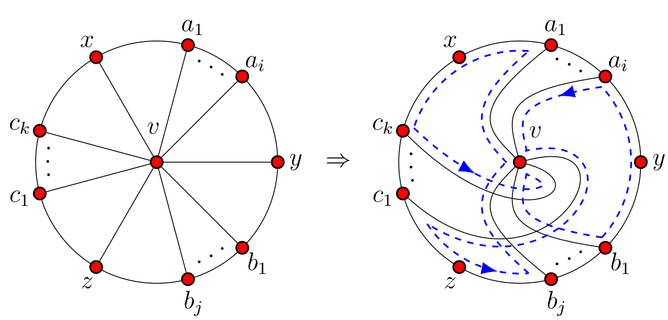

Figure 5: Altering the rotation at a vertex to increment the genus.

Inside of this 12-sided face, we try to add back the deleted edges and also add some new edges. The most immediate application of this construction simply connects x, y, and z:

Proposition 4.2. *If there exists a triangular embedding ϕ of $K_n - K_3$ then there exists an embedding of K_n in the surface of genus $I(n)$.*

Proof. One can check that the genus of ϕ is $I(n) - 1$. After applying Modification 4.1 and adding the chords (x, y), (y, z), and (x, z), we are left with the 5-sided faces $[0, b_1, y, x, c_k]$, $[0, c_1, z, y, a_i]$, and $[0, a_1, x, z, b_j]$, as in Figure 6. There are several options for adding back the edges $(0, x)$, $(0, y)$, and $(0, z)$ as chords. □

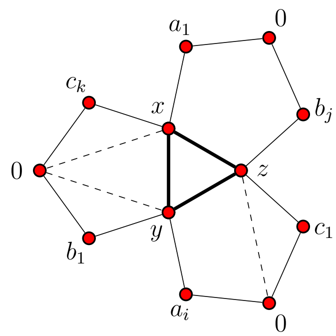

Figure 6: Adding in a K_3 with a handle with one possible way of restoring the deleted edges.

We note that this method alone cannot produce an embedding of type (6), for any choice of placements of the edges $(0, x)$, $(0, y)$, $(0, z)$. Thus, our families of current graphs need to satisfy additional properties so that the right sequence of chord exchanges can be applied to produce the desired 6-sided face.

5 Orientable minimum genus embeddings

5.1 Cases 2 and 5

Theorem 5.1. *For all $s \geq 1$, the graphs K_{12s+2} and K_{12s+5} have embeddings of type* (8)*,* (7, 4)*,* (6, 5)*,* (6, 4, 4)*,* (5, 5, 4)*,* (5, 4, 4, 4)*, and* (4, 4, 4, 4, 4)*.*

Proof. There exist triangular embeddings of $K_n - K_2$ for $n \equiv 2, 5$ (see Jungerman [10] and §9 of [17]). To add the missing edge, we excise a disk from two triangular faces and attach a handle between the resulting boundaries. Adding the missing edge along this handle, following Figure 7, creates an embedding of type (8). For $n \geq 14$, we apply chord exchanges to construct all the remaining embedding types:

(*Types* (7, 4)*,* (6, 4, 4)*,* (5, 5, 4)*, and* (4, 4, 4, 4, 4)). Since $s \geq 1$, both x and y have enough neighbors to allow us to find faces $[x, a, b]$ and $[y, c, d]$ such that a, b, c, d are all distinct vertices. After merging these two faces with a handle and adding the edge (x, y), the resulting 8-sided face is of the form $[x, a, b, x, y, c, d, y]$. Exchanging the following sets of chords yields the following embeddings:

type (7, 4):	(a, y);	type (6, 4, 4):	(a, d);
type (5, 5, 4):	(a, c);	type (5, 4, 4, 4):	$(a, d) \to (b, y)$;
type (4, 4, 4, 4, 4):	$(a, d), (b, c)$.		

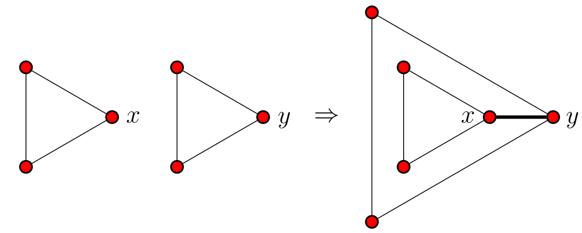

Figure 7: Adding an edge using a handle merges two triangular faces into an 8-sided face.

Note that for the type $(7,4)$ case, only one of the two possible chord exchanges yields the correct embedding type. The chord exchanges for type $(5,4,4,4)$ are illustrated in Figure 8(a).

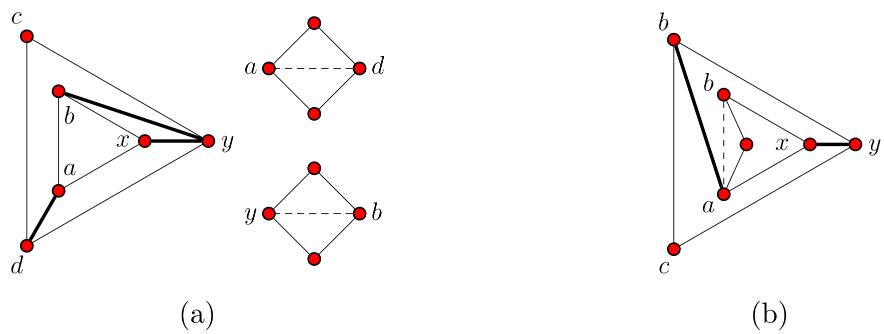

Figure 8: Finding embeddings of types $(5,4,4,4)$ (a) and $(6,5)$ (b).

(*Type* $(6,5)$). We assert that there exist faces $[x,a,b]$, $[y,b,c]$, where $a \neq c$. Vertex b has at least 13 neighbors, so without loss of generality, we may assume that the rotation at b is of the form

$$b. \quad \ldots \quad y \quad c \quad \ldots \quad a \quad x \quad \ldots,$$

i.e., that there are at least two other vertices in between y and x in the cyclic sequence. Hence, these triangles incident with b are the desired faces. If we delete the edge (a,b) and insert a handle between the resulting quadrilateral face and $[y,b,c]$, we can add the edges (x,y) and (a,b) in such a way that leaves one 6-sided and one 5-sided face, as in Figure 8(b). □

5.2 Case 9

We say that G_n is a *split-complete graph* if we can label its vertices $1,2,\ldots,n-1,x_0,x_1$ such that

- $1,\ldots,n-1$ are all pairwise adjacent, and
- the neighbors of x_0 and the neighbors of x_1 form a partition of $\{1,\ldots,n-1\}$.

Proposition 5.2. *If there exists a triangular embedding of a split-complete graph G_n, then there exists an embedding of type* (6) *of K_n.*

Proof. Add the missing edge x_0 and x_1 as we did for Cases 2 and 5. Note that the newly added edge (x_0, x_1) appears twice in the resulting 8-sided face. Locally contracting this edge leaves a 6-sided face, as in Figure 9. □

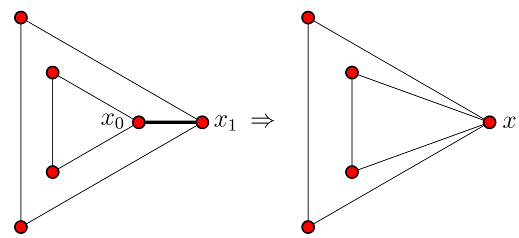

Figure 9: Attaching a handle to add an edge, and then contracting the edge to form a 6-sided face.

In §6.5 of Ringel [17], a triangular embedding of a split-complete graph G_{12s+9} is constructed for all $s \geq 0$, yielding the following:

Theorem 5.3. *For all $s \geq 0$, the graph K_{12s+9} has a nearly triangular minimum genus embedding.*

Corollary 5.4. *For all $s \geq 0$, the graph K_{12s+9} has embeddings of type (6), $(5,4)$, and $(4,4,4)$ of K_{12s+9}.*

5.3 Case 6

Our approach for much of Case 6 is via triangular embeddings of $K_n - P_3$, where P_3 is the path graph on three edges:

Lemma 5.5. *If there exists a triangular embedding of $K_n - P_3$, then there exists an embedding of type (6) of K_n.*

Proof. Suppose the missing edges are (a, b), (b, c), and (c, d). The edges (a, c) and (b, d) are in the graph, so there are triangular faces $[a, c, x]$ and $[d, b, x']$ for some possibly nondistinct vertices x and x'. Note that exactly one pair of faces incident with (a, c) and (b, d) have those specific orientations. With one handle, we can add back the missing edges following Figure 10, leaving the 6-sided face $[a, b, x', d, c, x]$. □

Theorem 5.6. *For all $s \geq 0$, the graph K_{12s+6} has a nearly triangular minimum genus embedding.*

Proof. For $s = 0$, such an embedding can be found by deleting a vertex from the triangular embedding of K_7 in the torus. For $s = 1$, Mayer [15] constructed a split-complete graph G_{18}, so applying Proposition 5.2 yields the desired embedding.

Gross [6] found triangular embeddings of $K_{12s+6} - P_3$ for $s \geq 3$. For the remaining case $s = 2$, we obtain such an embedding from the triangular embedding of $K_{30} - K_3$ given in §9.3 of Ringel [17] (and detailed in Appendix A) by deleting the edge $(6, x)$, exchanging the sequence of chords $(11, 16) \to (1, 26)$, and adding the edge (x, z). By applying Lemma 5.5 to each of these graphs, we obtain nearly triangular embeddings of K_{12s+6}, $s \geq 2$. □

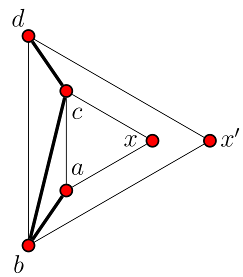

Figure 10: Adding a P_3 subgraph using a specific pair of faces to get a 6-sided face.

Corollary 5.7. *For all $s \geq 0$, the graph K_{12s+6} has embeddings of type* (6), $(5,4)$, *and* $(4,4,4)$.

5.4 Case 10

Theorem 5.8. *For all $s \geq 0$, the graph K_{12s+10} has a nearly triangular minimum genus embedding.*

Proof. For $s = 0$, we apply Lemma 5.5 to the triangular embedding of $K_{10} - P_3$ given in Appendix A.

We use the same current graph as Ringel [17, §2.3], except we reverse the rotation at the vertex adjacent to vortex z, as shown in Figure 11. Our solution to the additional adjacency problem hinges on the fact that the current $2s+1$ flowing into vortex x is twice that of $-(5s+3)$, the current flowing into vortex z. Let $c = -(5s+3) = 7s+4$. Then $2s+1 = 2c$ and $3s+2 = -3c$ in the group $\mathbb{Z}_{12s+7}$.

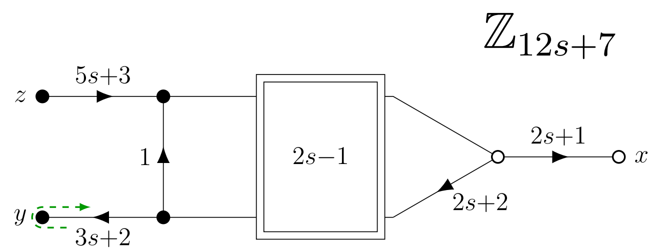

Figure 11: A current graph generating $K_{12s+10} - K_3$ with the pertinent currents marked.

After expressing the currents in Figure 11 in terms of the value c, the log of this current graph is of the form

$$-3c \quad y \quad 3c \quad 1 \quad c \quad z \quad -c \quad \ldots \quad -2c{-}1 \quad 2c \quad x \quad -2c \quad \ldots$$

After applying Modification 4.1 to vertices 0 and x, y, z, we obtain the 12-sided face

$$[x, 2c, 0, 3c, y, -3c, 0, -c, z, c, 0, -2c].$$

We exchange the chords (x, c) and $(x, 3c)$, in addition to the sequence

$$(2c, 3c) \to (2c{+}1, z) \to (c{+}1, 3c{+}1) \to (-c, x),$$

and add back the edges $(0, x)$, $(0, y)$, and $(0, z)$ according to Figure 12. The chord exchanges produce a 6-sided face by deleting three consecutive edges in the rotation at vertex x. $\square$

Corollary 5.9. *For all $s \geq 0$, the graph K_{12s+10} has embeddings of type (6), $(5, 4)$, and $(4, 4, 4)$.*

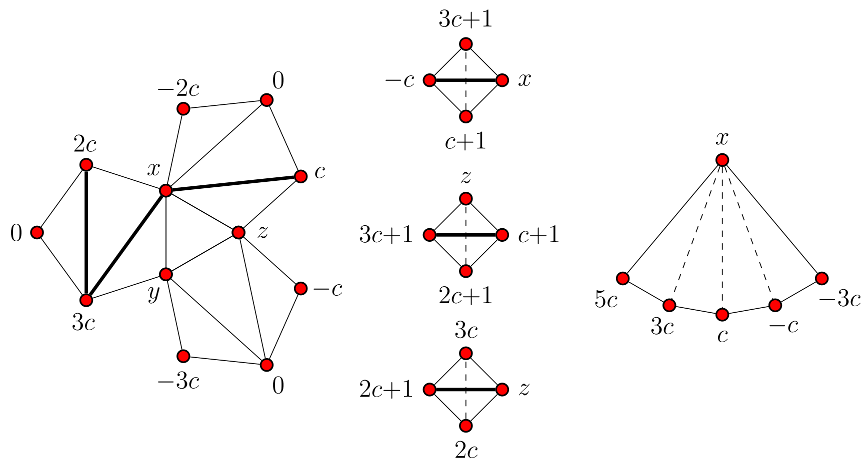

Figure 12: Obtaining a 6-sided face using chord exchanges.

5.5 Case 1

Theorem 5.10. *For all $s \geq 1$, the graph K_{12s+1} has a nearly triangular minimum genus embedding.*

Proof. The minimum genus embedding of K_{13} given by Ringel [17, p.82] already happens to be nearly triangular.

For $s = 2$, we use the current graph in Figure 13, which is isomorphic to one presented in Ringel [17, §6.3], and for $s \geq 3$, we use the family of current graphs in Figure 14. To interpret this drawing, replace the box in Figure 14(a) with the graph fragment in Figure 14(b). The elements 1, 3, and $6s-3$ are all generators of $\mathbb{Z}_{12s-2}$, so the vortices are all of type (T1) and the derived embeddings are triangular embeddings of $K_{12s+1} - K_3$.

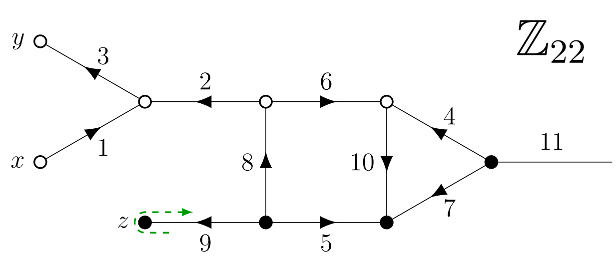

Figure 13: Gustin's current graph relabeled.

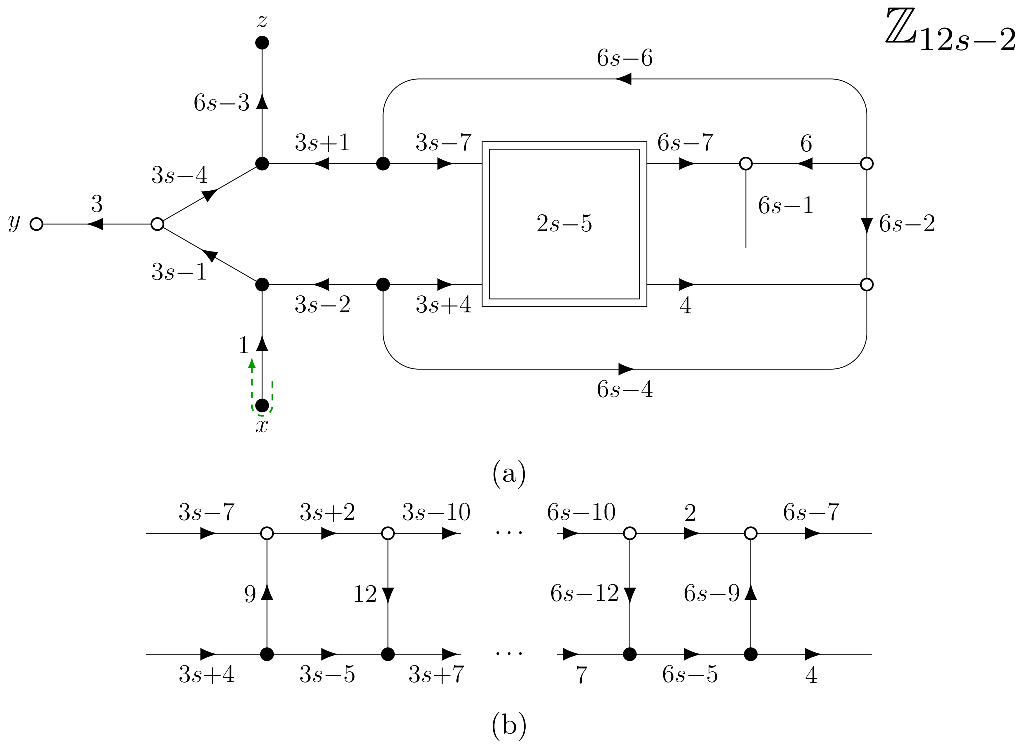

Figure 14: Current graphs for Case 1, $s \geq 3$.

For all these current graphs, the relative positions of the letters x, y, and z in the rotation at vertex 0 are the same, so applying Modification 4.1 on vertex 0 and vertices x, y, z produces the 12-sided face

$$[z, 6s{-}3, 0, -3, y, 3, 0, 1, x, -1, 0, 6s{+}1].$$

The sequences of chord exchanges

$$(x, 3) \to (2, 4) \to (5, 18) \to (z, 13)$$

for $s = 2$ and

$$(y, 6s{-}3) \to (6s{-}6, 6s) \to (0, 1)$$

for $s \geq 3$ removes one of the edges incident with the 12-sided face. If we add the remaining edges according to Figure 15, we obtain nearly triangular embeddings with one 6-sided face. □

Corollary 5.11. *For all $s \geq 1$, the graph K_{12s+1} has embeddings of type* (6)*,* (5, 4)*, and* (4, 4, 4)*.*

Remark. To the best of our knowledge, all previously published families of current graphs for orientable triangular embeddings of $K_{12s+1} - K_3$ split into two subfamilies depending on the parity of s. Our current graphs in Figure 14 form a solution for all $s \geq 3$ irrespective of parity. Another such family of current graphs is presented in Appendix B that includes the $s = 2$ case, but we were unable to use it to prove Theorem 5.10.

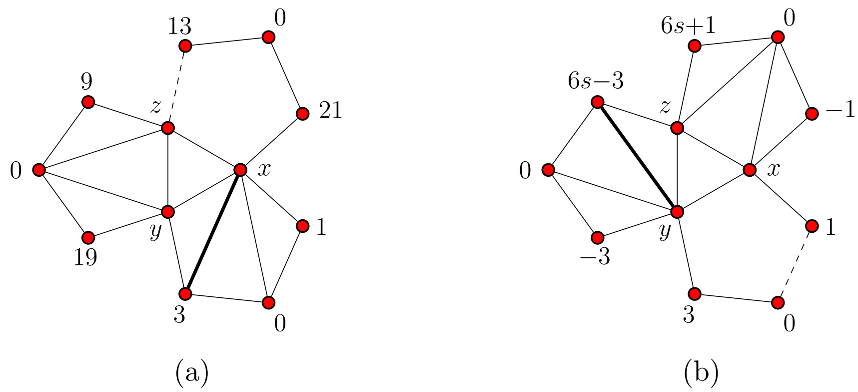

Figure 15: The end result of the Case 1 additional adjacency for $s = 2$ (a) and $s \geq 3$ (b).

5.6 Case 8

Theorem 5.12. *For all $s \geq 1$, the graph K_{12s+8} has a nearly triangular minimum genus embedding.*

Proof. We use a variety of current graphs with current group $\mathbb{Z}_{12s+6}$: for $s = 1$, the index 3 current graph described in Appendix A, for $s = 2$, the current graph of Ringel and Youngs [20] in Figure 16, and for $s \geq 3$, the new family of current graphs in Figure 17.

In all their derived embeddings, there is vertex x incident with only the numbered vertices, and two vertices y_0 and y_1 incident with only the even and odd numbered vertices, respectively. With one handle, we wish to connect the vertices x, y_0, and y_1 to produce a nearly triangular embedding. If we delete the edge $(6s-1, x)$, we can exchange the sequence of chords $(6s, 6s-2) \to (0, 6s+4)$ so that the edge $(12s+1, y_0)$ can be added. Now, vertex $12s+1$ is incident with all the lettered vertices, as in Figure 18(a).

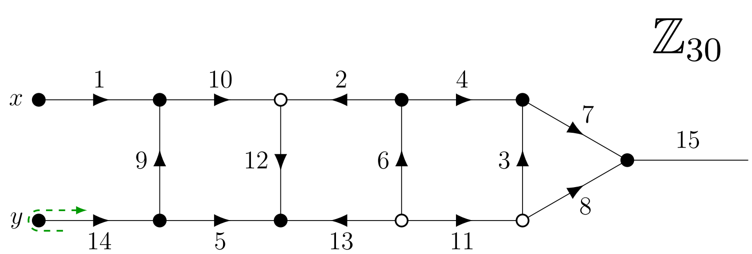

Figure 16: The current graph of Ringel and Youngs [20] for K_{32}.

If we apply Modification 4.1 to vertex $12s+1$ and neighbors y_0, y_1, x, we obtain the 12-sided face

$$[y_0, 6s+4, 12s+1, 6s-3, y_1, 6s-1, 12s+1, 12s+2, x, 12s, 12s+1, 0].$$

Adding the edge (y_0, y_1) in this face and contracting it to make a new vertex y yields one 4-sided face and one 8-sided face, and the remaining edges (x, y), $(y, 12s+1)$, $(x, 12s+1)$, and $(x, 6s-1)$ can be added back in, according to Figure 18(b), to produce an embedding of type (5). □

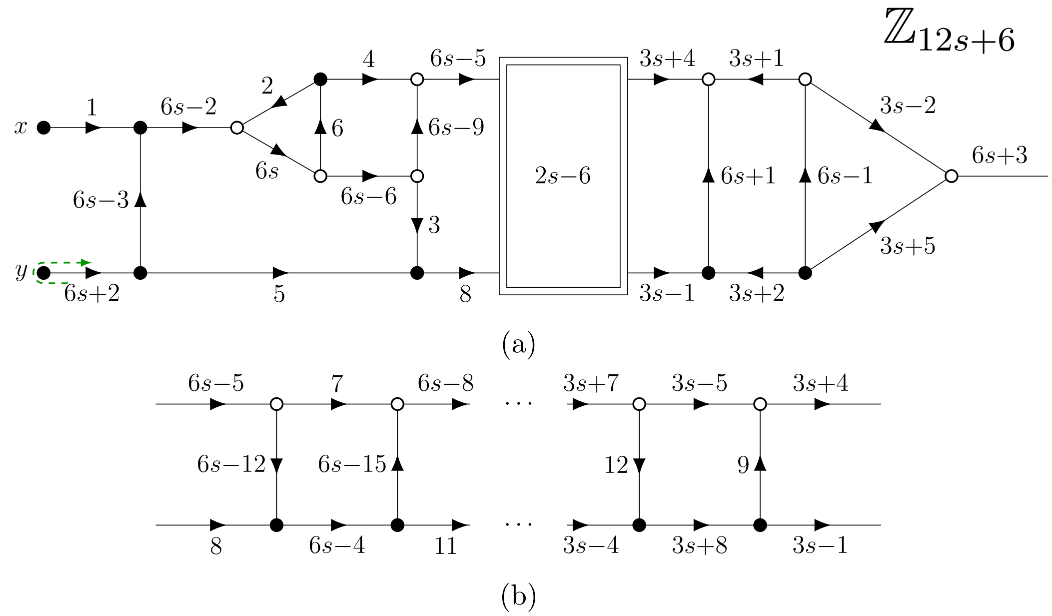

Figure 17: A new family of current graphs for $s \geq 3$.

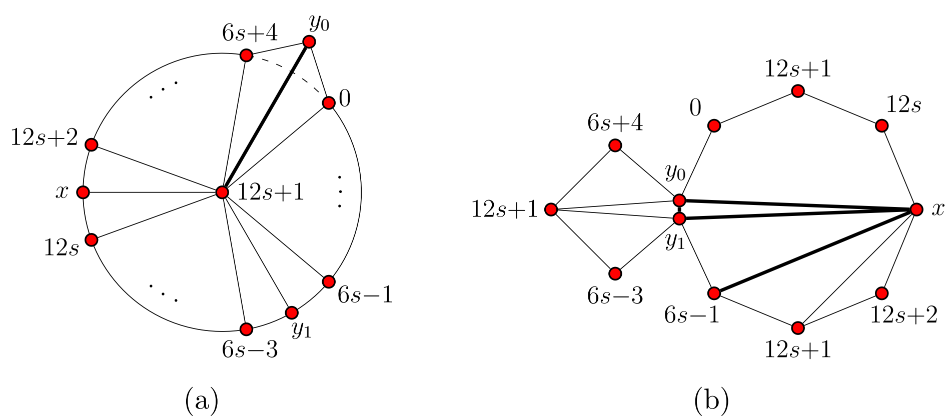

Figure 18: An initial modification near vertex $12s{+}1$ (a) and using one handle to connect all lettered vertices and to replace the missing edge $(x, 6s{-}1)$ (b).

Corollary 5.13. *For all $s \geq 1$, the graph K_{12s+8} has embeddings of type* (5) *and* (4, 4).

Remark. The additional adjacency solution presented here can also be applied to the family of current graphs of Ringel and Youngs [20] for $s \geq 4$. Our family of current graphs, while slightly more complicated in terms of the underlying graph, benefits from a significantly simpler current assignment. In addition, our solution handles the odd and even s cases simultaneously, and it extends downwards to $s = 3$, for which Ringel and Youngs needed a special embedding that cannot be easily modified into a nearly triangular embedding.

5.7 Case 11

Theorem 5.14. *For all $s \geq 0$, the graph K_{12s+11} has a nearly triangular minimum genus embedding.*

Proof. Ringel [17, §5.2] describes the minimum genus embeddings of K_{11} and K_{23} of Mayer [15] with doubled edges deleted and each nontriangular face subdivided with a new vertex. Deleting the subdivision vertices in both embeddings leaves two quadrilateral faces. We obtain nearly triangular embeddings each with one 5-sided face by exchanging the chords $(1, 8)$ and $(11, 8)$ in the embedding of K_{11} and exchanging the sequences of chords

$$(10, 16) \to (6, 20) \quad \text{and} \quad (8, 22) \to (9, 14) \to (5, 6)$$

in the embedding of K_{23}.

For the general case $s \geq 2$, tracing through the construction of Ringel and Youngs [19] in primal form, as done in Theorem 8 of Korzhik and Voss [13], yields a minimum genus embedding of K_{12s+11} with two quadrilateral faces $[0, 6s+5, x, y]$ and $[0, a, b, 4]$ and the triangular face $[0, b, x]$ (see Figure 14 of [13]). Exchanging the chords $(0, x)$ and $(0, b)$ yields a nearly triangular embedding. □

Corollary 5.15. *For all $s \geq 0$, the graph K_{12s+11} has embeddings of type* (5) *and* (4, 4)*.*

6 Nonorientable embeddings

Let N_k denote the nonorientable surface of genus k, a sphere with k crosscaps, and let $\overline{\gamma}(G)$ be the *minimum nonorientable genus* of G. Analogously, for a nonorientable embedding $\phi : G \to N_k$, we have the *nonorientable* Euler polyhedral equation

$$|V(G)| - |E(G)| + |F(G, \phi)| = 2 - k$$

and the *nonorientable* genus formula for the complete graphs:

$$\overline{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil, n \geq 3, n \neq 7.$$

The discrepancies with the orientable versions are due to the fact that one handle in a nonorientable surface is homeomorphic to two crosscaps. The lone exception $n = 7$ is due to Franklin [3], who showed that K_7 cannot embed in N_2, the Klein bottle.

Because crosscaps are "half of a handle," we can obtain nonorientable triangular embeddings for some complete graphs that cannot triangulate an orientable surface. For $n \equiv 1, 6, 9, 10 \pmod{12}$, we showed that there were orientable embeddings of type (6) of K_n, but these graphs actually have nonorientable triangular embeddings (see Ringel [17]). Similarly, we used a handle to add the missing edge to a triangular embedding of $K_n - K_2$ for $n \equiv 2, 5 \pmod{12}$, but actually a crosscap suffices. We show, using only known constructions, that all the predicted embedding types for minimum nonorientable genus embeddings exist:

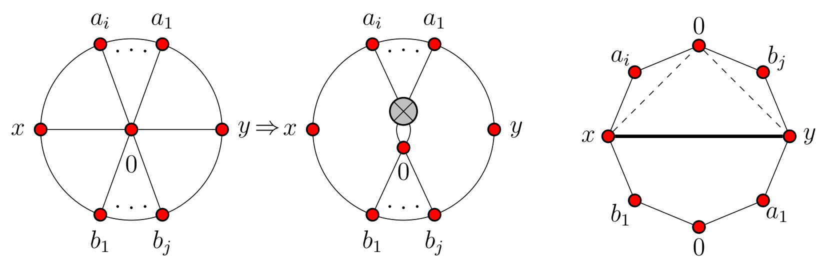

Figure 19: Adding one edge with a crosscap.

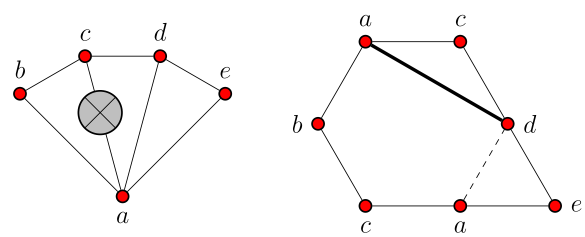

Figure 20: Some simple modifications to the triangular embedding of K_7.

Theorem 6.1. *For* $n \equiv 2, 5, 8, 11 \pmod{12}$*,* $n \geq 5$*, the complete graph* K_n *has embeddings of types* (5) *and* $(4, 4)$ *in nonorientable surfaces.* K_7 *has minimum genus embeddings of types* (6)*,* $(5, 4)$*, and* $(4, 4)$ *in* N_3*.*

Proof. Like in the orientable case, it suffices to find the nearly triangular embedding. That is, a nonorientable analogue of Proposition 2.7 also holds for embeddings of type (5). Korzhik [11] shows that the standard proof of Case 8 of the nonorientable Map Color Theorem (see also Ringel [17, §8.3]) yields an embedding of type (5) of K_{12s+8}, $s \geq 0$. For the remaining Cases, there are several known ways [10, 11, 17] of constructing triangular embeddings of $K_n - K_2$, $n \equiv 2, 5, 11$ in possibly nonorientable surfaces. The interpretation by Korzhik [11] of the standard construction for adding an edge with one crosscap results in an embedding of type $(4, 4)$, but a small modification produces a nearly triangular embedding:

Let x and y be the two nonadjacent vertices of $K_n - K_2$, and let 0 be another vertex adjacent to both x and y. As seen in Figure 19, by deleting the edges $(0, x)$ and $(0, y)$ and passing some of the other edges incident with vertex 0 through a crosscap, we obtain an 8-sided face incident with x, y, and 0. After adding the chord (x, y), there is a choice for adding back the removed edges $(0, x)$ and $(0, y)$ that yields an embedding of type (5).

The exceptional graph K_7 has a triangular embedding in S_1, but not in N_2. Let a be any vertex in this toroidal embedding and suppose its rotation is of the form

$$a. \quad \ldots \quad b \quad c \quad d \quad e \quad \ldots$$

Then, as in Figure 20, we can add a crosscap along the edge (a, c) to obtain a cellular embedding of type (6) in N_3. Unlike in Proposition 2.7, where we explicitly invoked orientability, the resulting hexagonal face $[c, a, b, c, a, d]$ has two pairs of repeated vertices. However, the chord exchange on the right of Figure 20 removes one of those repetitions, allowing us to use additional chord exchanges to obtain the embeddings of type $(5, 4)$ and $(4, 4, 4)$. □

7 Maximum genus embeddings

The *(orientable) maximum genus* $\gamma_M(G)$ is the largest integer g such that G has a cellular embedding in S_g. Archdeacon and Craft [1] also ask if the complete graphs have nearly triangular maximum genus embeddings. The complete graphs K_n are known to be *upper-embeddable* (see Nordhaus *et al.* [16]), meaning they have embeddings with either one or two faces, depending on the parity of $|E(K_n)| - |V(K_n)|$. One-face embeddings are already trivially nearly triangular, so we need a construction just for two-face embeddings.

A special case of Xuong's characterization [23] of maximum genus states that a graph G is upper-embeddable if and only if there is a spanning tree T such that $G - T$ has at most one component with an odd number of edges. To construct a one- or two-face embedding, the edges of $G-T$ are partitioned into pairs such that the edges of each pair share a vertex. Starting with an arbitrary embedding of the spanning tree T in the plane (which has one face), we add the pairs one by one, as in Figure 21. After each addition, the resulting embedding still has one face. If there is an edge left over (i.e. one of the edges of the odd-sized component), it is added arbitrarily into the embedding, resulting in a two-face embedding.

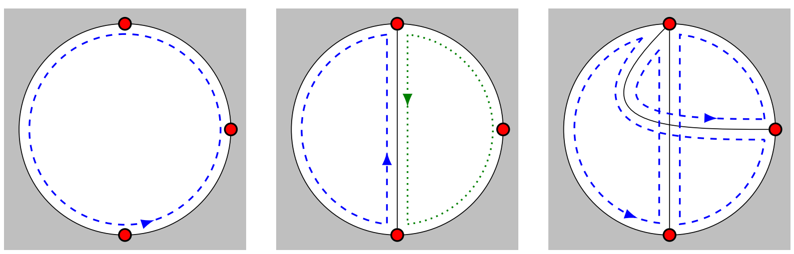

Figure 21: Adding two edges to a one-face embedding.

We note that the final embedding of G restricted to T is the same as the original embedding of T that we started with. This observation is enough for constructing a nearly triangular two-face embedding.

Proposition 7.1. *For $n \equiv 0, 3 \pmod 4$, there exists a two-face embedding of K_n where one of the faces is a triangle.*

Proof. Label the vertices $1, \ldots, n$. Delete the edge $(2, 3)$ and let the spanning tree T be all the edges incident with vertex 1. Then, the graph $(K_n - (2, 3)) - T$ is connected and has an even number of edges. Let the rotation at vertex 1 in the embedding of T be

$$1.\quad 2\quad 3\quad \ldots\quad n{-}1\quad n.$$

Adding all pairs of edges in the manner described above preserves the rotation at vertex 1, resulting in an embedding with one face of the form $[\ldots 2, 1, 3 \ldots]$. We then insert the edge $(2, 3)$ into the embedding to obtain an embedding of K_n with one triangular face $[2, 1, 3]$ and one long nontriangular face. □

Finally, the problem for the nonorientable maximum genus $\overline{\gamma}_M$ is the simplest of them all. A well-known result (see, e.g., Stahl [21]) states that *every* connected graph with a cycle has a one-face embedding in a nonorientable surface, so there is nothing to prove. The main idea behind its proof can be extended to the following "interpolation" theorem:

Proposition 7.2. *For every nonorientable surface N_k, where*

$$k \in [\overline{\gamma}(K_n), \overline{\gamma}_M(K_n)],$$

K_n has a nearly triangular embedding in N_k.

Proof. Let ϕ be a nearly triangular nonorientable minimum genus embedding of K_n. Let f be the nontriangular face—if the embedding is triangular, let f be an arbitrary face. If ϕ is not already a one-face embedding, then there exists an edge incident with f and a different face. Adding a crosscap on this edge merges the two faces, incrementing the genus of the embedding. Applying this procedure repeatedly, starting at a minimum genus embedding and ending at a one-face embedding, yields the desired result. □

8 Concluding Remarks

We resolved the question of Archdeacon and Craft [1], describing the complete graphs with a nearly triangular minimum genus embedding. Interest in these types of embeddings originated in searching for nonisomorphic minimum genus embeddings of the complete graph. While Korzhik and Voss [13] found exponential families of embeddings for the complete graphs that do not triangulate a surface, their approach only looked at a single face distribution per graph. Can the results presented here be used to construct exponential families for the other face distributions?

The techniques of Korzhik and Voss [13] are extendable to Cases 1, 8, 9, 10, 11 by modifying the rotations at the vertices hidden by the box in Figure 4. They also construct exponential families of nearly triangular embeddings for Case 5, so the same construction with chord exchanges should produce exponential families for the other face distributions. The situation is uncertain for Cases 2 and 6—Korzhik and Voss had constructions for these Cases, but they used different current graphs that have not been shown to yield nearly triangular embeddings.

For Case 6, we found a solution using a result of Gross [6] for the related problem of finding triangular embeddings of "nearly complete" graphs. Do other nearly complete graphs belonging to the other Cases have similar results? Case 9 seems to be the most accessible, since triangular embeddings of $K_{12s+9} - K_3$ can be constructed in a similar way (see Youngs [24]) as the embeddings of $K_{12s+6} - K_3$ used by Gross.

In Cases 2, 5, and 9, we constructed nearly triangular embeddings where the nontriangular faces are not simple. Can the constructions be modified in some way to produce nearly triangular embeddings with simple faces? We suspect that this problem will be difficult for the first two of those Cases.

The theory of index 3 current graphs allowed us to extend the additional adjacency approach of Ringel and Youngs [20] to handle K_{20}, which previously needed an asymmetric embedding. Two interesting directions would be generalizing this solution to all of Case 8, or using the same approach for Case 11.

9 Acknowledgements

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A Embeddings of small graphs

In this appendix, we describe triangular embeddings that were not produced using index 1 current graphs. The following embedding of $K_{10} - P_3$ is used in the proof of Theorem 5.8:

0.	2	6	5	7	4	3	8	9	
1.	3	5	6	9	4	8	7		
2.	0	9	7	5	8	4	6		
3.	0	4	5	1	7	9	6	8	
4.	0	7	6	2	8	1	9	5	3
5.	0	6	1	3	4	9	8	2	7
6.	0	2	4	7	8	3	9	1	5
7.	0	5	2	9	3	1	8	6	4
8.	0	3	6	7	1	4	2	5	9
9.	0	8	5	4	1	6	3	7	2

We describe two embeddings generated by index 3 current graphs. For more details, see §9 of Ringel [17]. Instead of one log, there are now three, each corresponding to one of the faces of the embedding. Rotations of numbered vertices are generated in a similar manner, except that for vertex $k \in \mathbb{Z}_{3n}$, we add k to all the numbered elements in the log labeled $[k \bmod 3]$. As with vortices in index 1 current graphs, lettered vertices signify long nontriangular faces that have been subdivided with new vertices. The current graph depicted in Figure 9.12 of Ringel [17] has the logs

[0].	26	15	16	24	8	6	25	4	7	22	9	18	13	z	...
	14	1	12	11	3	19	21	2	23	20	5	x	10	y	17
[1].	26	13	9	18	4	y	17	x	25	z	14	5	12	22	...
	10	15	20	24	16	19	3	23	21	1	7	6	2	8	11
[2].	1	18	9	10	y	23	14	13	z	2	x	22	15	5	...
	17	12	7	3	11	8	24	4	6	26	20	21	25	19	16

with current group $\mathbb{Z}_{27}$. The rotation at, e.g., vertex x reads:

x.	0	5	7	24	2	4	21	26	1	18	23	25	15	...
	20	22	12	17	19	9	14	16	6	11	13	3	8	10

For Case 8, $s = 1$, we made use of another index 3 current graph, depicted in Figure 22. Its logs are

[0].	5	3	14	11	15	2	7	12	16	4	6	13	10	y	8	9	17	x	1
[1].	7	9	16	10	y	8	3	4	17	x	1	15	5	11	13	6	2	14	12
[2].	13	14	17	x	1	10	y	8	6	9	15	4	7	3	12	2	11	5	16

B A new solution for Case 1

In Section 5.5, we found nearly triangular embeddings of K_{12s+1} for $s \geq 3$ using a single family of current graphs with a simple current assignment. In fact, there even exist families of current graphs for $K_{12s+1} - K_3$ that include the $s = 2$ case as well, such as the one in

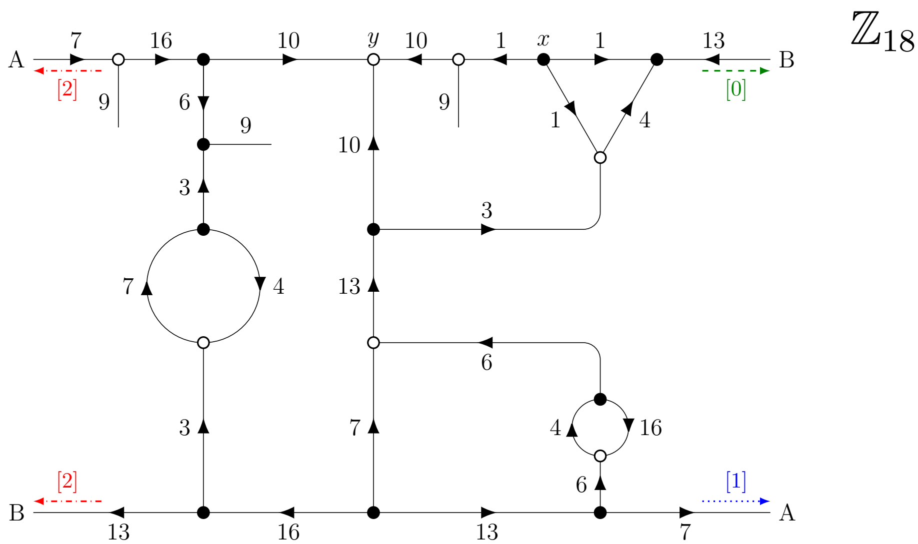

Figure 22: An index 3 current graph for Case 8, $s = 1$, where capital letters indicate identifications.

Figure 23. This is the simplest known proof of Case 1 for $s \geq 2$ of the original Map Color Theorem, and as remarked by Ringel [17, p.96], there cannot exist an index 1 current graph with three vortices of type (T1) for $s = 1$ because $\mathbb{Z}_{10}$ does not have enough generators.

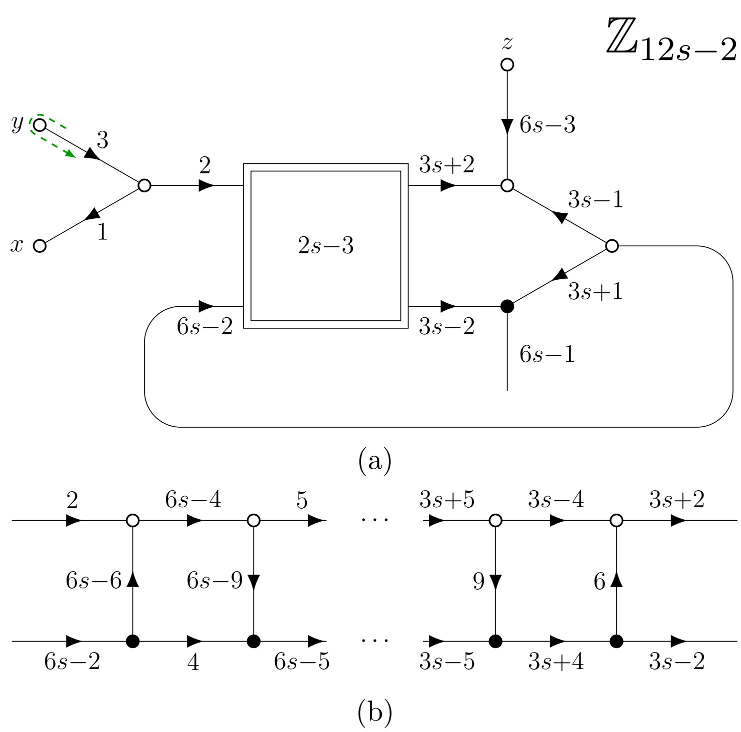

Figure 23: Current graphs producing triangular embeddings of $K_{12s+1} - K_3$ for all $s \geq 2$.