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A simple construction for orientable triangular embeddings of the complete graphs on 12s vertices

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ABSTRACT

We present a family of index 1 abelian current graphs whose derived embeddings can be modified into triangular embeddings of K_{12s} for $s \geq 3$. Our construction is significantly simpler than previous methods for finding genus embeddings of K_{12s}, which utilized either large index or nonabelian groups.

1. Introduction

Showing that the complete graph K_n can be embedded in the orientable surface of genus

$$\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

proves the Map Color Theorem for orientable surfaces of positive genus [6]. The case of the graphs K_{12s} was of particular difficulty, as its original solution utilized *nonabelian* current groups. Namely, Terry et al. [7] employed the representation theory of finite fields to label the arcs of a current graph with elements from a nonabelian group of order 12s.

To avoid the difficulties that arise from using nonabelian groups, Pengelley and Jungerman [5] announced a solution using index 4 current graphs with the cyclic group $\mathbb{Z}_{12s}$, but gave details only for $s = 1$ and $s \equiv 0 \pmod 8$. Korzhik [3] improved on their method, giving four families of current graphs, one for each residue s mod 4, for $s = 4$ and $s \geq 6$. Unfortunately, the aforementioned current graphs in this high index regime are even more complicated than those of Terry et al. [7], despite the much simpler group involved.

Mahnke [4] proved that there do not exist index 1 current graphs with abelian current groups that directly generate orientable triangular embeddings of K_{12s}. We circumvent this barrier by using index 1 current graphs with cyclic groups $\mathbb{Z}_{12s-4}$ to first generate certain orientable embeddings of K_{12s-4}, which then can be modified into the desired orientable triangular embeddings of K_{12s}.

2. Graph embeddings generated by current graphs

For more information on topological graph theory, especially a formal treatment of current graphs, see Gross and Tucker [1]. Proofs of correctness for the combinatorial techniques described here can be found in Ringel [6]. In this paper,

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we only consider embeddings in *orientable* surfaces—the analogous statement for the nonorientable genus of K_{12s} is swiftly handled in §8.2 of Ringel [6].

A consequence of Euler's formula is that the genus of a simple graph G is at least

$$\frac{|E(G)| - 3|V(G)| + 6}{6},$$

and this value is attained by a *triangular embedding* of G, i.e., an embedding where every face is triangular. Our main result is thus a new construction for triangular embeddings of K_{12s}, which are embeddings in the surface of genus $(4s-1)(3s-1)$. The expression for the genus of a triangular embedding also implies, roughly speaking, that a handle corresponds to six edges in such an embedding. This relationship is crucial for so-called "additional adjacency" solutions in the proof of the Map Color Theorem, where a triangular embedding of a nearly complete graph is augmented using handles to obtain a genus embedding of a complete graph. Ringel et al. were successful in finding *non-triangular* genus embeddings in this manner, but in the present paper we apply an additional adjacency step that terminates in another triangular embedding.

A cellular embedding of a simple graph on a surface can be described combinatorially by a *rotation system*, where each vertex v is assigned a *rotation*, a cyclic permutation of its neighbors. The faces of the embedding can be traced out from the rotation system, but here, only the following specialization for triangular embeddings is necessary:

Proposition 2.1. *An embedding is triangular if and only if the corresponding rotation system satisfies the following property: for all vertices i, j, k, if the rotation at vertex i is of the form*

$$i. \ \ldots \ j \ k \ \ldots,$$

then the rotation at j is of the form

$$j. \ \ldots \ k \ i \ \ldots$$

A *current graph* is a directed graph D embedded in a surface, whose arcs are labeled using elements from a group Γ. We refer to these labels as *currents* and Γ as the *current group*. The *index* of a current graph is the number of faces in its embedding. For the remainder of the paper, except in the Appendix, we only consider index 1 current graphs and take Γ to be the cyclic groups $\mathbb{Z}_{12s-4}$. Let the *excess* of a vertex be the sum of its incoming currents minus the sum of its outgoing currents. A vertex satisfies *Kirchhoff's current law* (KCL) if its excess is zero.

All aforementioned work on K_{12s}, including Mahnke's nonexistence result [4], deals with current graphs where the derived graphs are exactly K_{12s}. Our departure from previous approaches is that we first construct a triangular embedding of $K_{12s}-K_4$ using an index 1 current graph, then use an additional handle to add the six missing edges. Triangular embeddings of $K_{12s}-K_4$ were known to Jungerman and Ringel [2], but for our purposes, we construct a different family of current graphs that satisfy additional properties. These current graphs exist for $s \geq 3$—to handle the smaller cases, we appeal to a well-known index 4 current graph (see Ringel [6, p. 82]) for $s = 1$, and in the Appendix, we describe another index 4 current graph for $s = 2$. Pengelley and Jungerman [5] and Korzhik [3] announced an index 4 solution for this small case, but no such current graph appears in either paper or elsewhere in the literature.

In order to obtain a triangular embedding of $K_{12s} - K_4$, we use an index 1 current graph with current group $\mathbb{Z}_{12s-4}$ to generate a particular embedding of K_{12s-4}. All of its faces are triangular except for four $(12s-4)$-gonal faces whose boundary walks are Hamiltonian cycles. These $(12s-4)$-gonal faces are induced by *vortices* in the current graph, i.e., vertices where KCL is not satisfied. In our drawings, we label each incident face corner of a vortex with a letter. To obtain a triangular embedding of $K_{12s} - K_4$, we subdivide each of the $(12s-4)$-gonal faces using new vertices labeled with those letters.

Our aforementioned current graphs with current groups $\mathbb{Z}_{12s-4}$ satisfy the following "construction principles", which we paraphrase from Jungerman and Ringel [2]:

- (C1) Every vertex has degree 1, 2, or 3.
- (C2) The embedding of D has one face, i.e. the current graph is of index 1.
- (C3) For every element $\gamma \in \mathbb{Z}_{12s-4} \setminus \{0\}$, exactly one of γ or $-\gamma$ appears as a current on exactly one arc of D.
- (C4) KCL holds at every vertex of degree 3.
- (C5) There is exactly one vertex of degree 1 with excess $6s-2$, the order 2 element of $\mathbb{Z}_{12s-4}$. All other vertices of degree 1 have an excess that generates $\mathbb{Z}_{12s-4}$.
- (C6) A vertex of degree 2 is incident with two arcs with odd currents, and its excess generates the subgroup of even elements of $\mathbb{Z}_{12s-4}$.

The one face boundary can be expressed as a cyclic sequence that alternates between arcs and face corners. Its *log* is obtained by replacing arcs with their currents, and corners with their vortex letters, if any. If an arc labeled γ is traversed in the same direction as its orientation, the arc is replaced by γ; if it is traversed in the reverse direction, the arc is replaced by $-\gamma$. By (C5), the order 2 element appears twice consecutively in the log, which by convention, we condense into one instance. The standard interpretation is that the derived graph actually has a "doubled 1-factor" which is then suppressed—for this

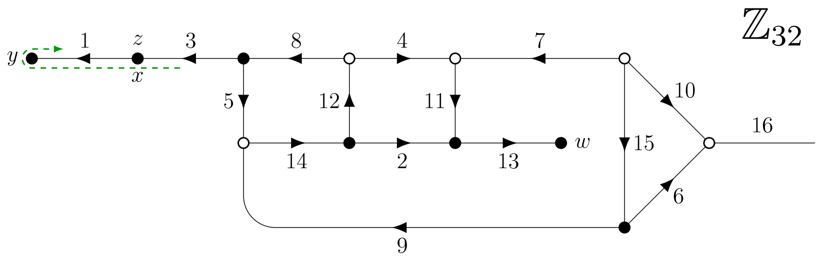

Fig. 1. An index 1 current graph with current group $\mathbb{Z}_{32}$ generating a triangular embedding of $K_{36} - K_4$. Solid and hollow vertices represent clockwise and counterclockwise rotations, respectively.

reason, the degree 1 endvertex of the arc labeled with $6s-2 \in \mathbb{Z}_{12s-4}$ is not considered a vortex and hence has no letter assigned to it.

The log of a current graph describes a symmetric embedding of a certain derived graph, whose vertices are the elements of $\mathbb{Z}_{12s-4}$ and the vortex letters. For example, the log of the current graph in Fig. 1 is the cyclic sequence

$$\begin{array}{ccccccccccccccccc} 3 & x & 1 & y & 31 & z & 29 & 24 & 20 & 2 & 21 & 25 & 15 & 6 & 16 & 22 & 7 & \dots \\ 28 & 8 & 5 & 23 & 17 & 10 & 26 & 9 & 14 & 12 & 4 & 11 & 13 & w & 19 & 30 & 18 & 27. \end{array} \tag{1}$$

The log describes the rotation at vertex 0 (i.e. the identity element of $\mathbb{Z}_{12s-4}$) and furthermore determines the rest of the rotation system. We obtain the rotation at vertex $k \in \mathbb{Z}_{12s-4}$ by adding k to each of the elements of $\mathbb{Z}_{12s-4}$ in the log. For the vortices of degree 1, we leave their letters unchanged, but for vortices of degree 2, such as the one corresponding to the letters x and z in this example, we switch the letters' positions if k is odd. A partial picture of the rotation system would therefore look like

$$\begin{array}{llllllllllllll} 0. & 3 & x & 1 & y & 31 & z & 29 & 24 & 20 & 2 & 21 & 25 & \cdots \\ 1. & 4 & z & 2 & y & 0 & x & 30 & 25 & 21 & 3 & 22 & 26 \\ 2. & 5 & x & 3 & y & 1 & z & 31 & 26 & 22 & 4 & 23 & 27 \\ 3. & 6 & z & 4 & y & 2 & x & 0 & 27 & 23 & 5 & 24 & 28 \\ 4. & 7 & x & 5 & y & 3 & z & 1 & 28 & 24 & 6 & 25 & 29 \\ \vdots & & & & & & & & & & & & & \ddots \end{array}$$

The rotations at the lettered vertices are "manufactured" such that the embedding near these vertices is triangular, which we do with the help of Proposition 2.1. For example, the rotation at vertex x is of the following form:

$$x. \quad \dots \quad 31 \quad 30 \quad 1 \quad 0 \quad 3 \quad 2 \quad 5 \quad 4 \quad \dots$$

Because the current graph satisfies the construction principles, the entire embedding is triangular, and the rotations at the lettered vertices form proper cyclic permutations of all the elements of $\mathbb{Z}_{32}$. This is a triangular embedding of $K_{36} - K_4$, as the only missing adjacencies are those between lettered vertices. We will show in the next section how we can add the remaining edges using one handle to get a triangular embedding of K_{36} as desired.

3. A new construction for triangular embeddings of K_{12s}, $s \geq 3$

For $s \geq 4$, consider the current graphs in Fig. 2 with current groups $\mathbb{Z}_{12s-4}$. By examining their logs near the vortices and the curved arcs in part (a), we find that the rotation at vertex 0 is of the form

$$0. \quad x \quad 1 \quad y \quad 12s{-}5 \quad z \quad \dots \quad 6s{+}2 \quad 2 \quad 6s{+}3 \quad \dots \quad 6s{-}2 \quad 6 \quad 6s{-}3 \quad \dots \quad 6s{-}5 \quad w \quad \dots$$

For $s = 3$, the rotation (1) of the current graph in Fig. 1 differs from this general form in that the two elements adjacent to 6 are swapped. That is, it is of the form

$$0. \quad x \quad 1 \quad y \quad 12s{-}5 \quad z \quad \dots \quad 6s{+}2 \quad 2 \quad 6s{+}3 \quad \dots \quad 6s{-}3 \quad 6 \quad 6s{-}2 \quad \dots \quad 6s{-}5 \quad w \quad \dots$$

Fig. 3 illustrates how this partial information allows us to add the missing edges using edge flips and a handle. In particular, adding the edge (x, y) at the cost of the edge $(0, 1)$ allows us to install a handle that merges three faces containing the four lettered vertices. We then add the missing edges near this handle or via sequences of edge flips. The minor discrepancy between the logs for $s = 3$ and $s \geq 4$ manifests in one of the quadrilaterals in Fig. 3—that quadrilateral is merely mirrored for $s = 3$, so the corresponding edge flip is still permissible and the rest of the additional adjacency solution is identical. The resulting embeddings of K_{12s} are triangular for all $s \geq 3$, completing the construction.

Remark. The arcs labeled 9 and $6s-12$ in Fig. 2(a) extend the arithmetic sequence in Fig. 2(b), but unfortunately the rotations assigned to their endpoints must differ from the pattern in Fig. 2(b). Thus, generalizing this family of current graphs to the $s = 3$ case is impossible.

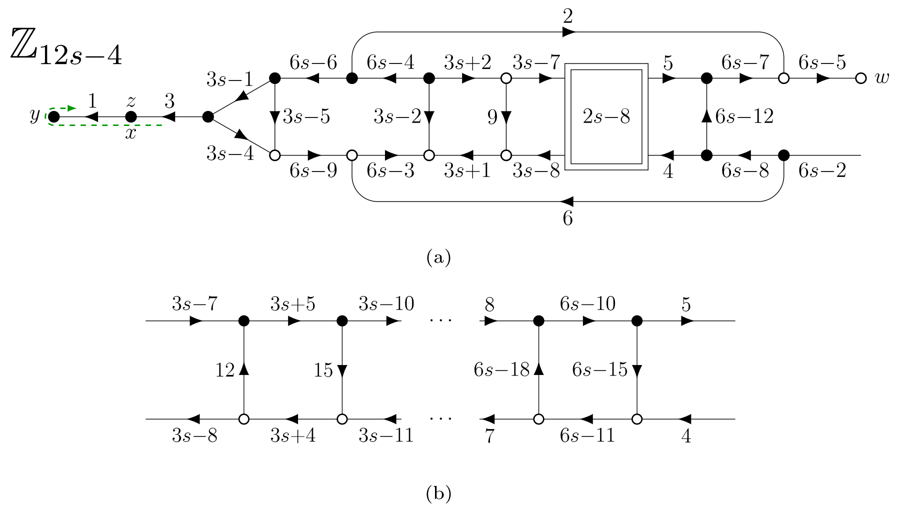

Fig. 2. The fixed part of the current graph (a) contains the salient currents for adding one handle, and the simple ladder (b) inside the box. The label on the box indicates the number of "rungs" in the ladder, whose orientations alternate and whose currents form an arithmetic sequence.

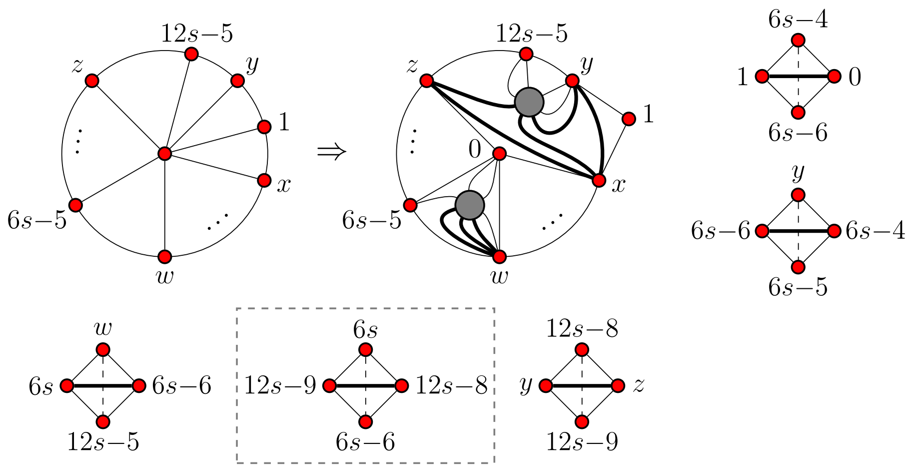

Fig. 3. The embedding near vertex 0 is augmented using a handle, which is represented by excising two disks and identifying their boundaries. For the edge flips, the dashed edges are replaced by the thick solid edges. The quadrilateral inside the box has the reverse orientation for $s = 3$.

Appendix. A triangular embedding of K_{24} using cyclic groups

Fig. A.1 details an index 4 current graph that generates a triangular embedding of K_{24} using the cyclic group $\mathbb{Z}_{24}$. The logs of this current graph are

[0].	19	16	4	1	21	20	12	8	3	5	2	13	11	22	10	17	7	14	6	15	9	18	23
[1].	5	8	20	23	3	4	12	16	21	19	22	11	13	2	14	7	17	10	18	9	15	6	1
[2].	19	20	21	1	4	23	5	6	15	9	18	8	16	10	17	7	14	12	2	13	11	22	3
[3].	5	4	3	23	20	1	19	18	9	15	6	16	8	14	7	17	10	12	22	11	13	2	21

To generate the rotation at vertex $k \in \mathbb{Z}_{24}$, take the log $[k \bmod 4]$ and add k to each element. One can verify that the embedding is triangular using Proposition 2.1. For more information on index 4 current graphs and the additional difficulties they incur, see Pengelley and Jungerman [5] or Korzhik [3].

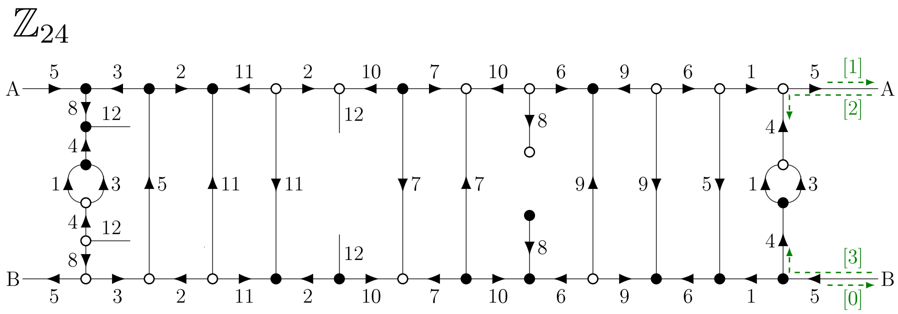

Fig. A.1. This index 4 current graph, which generates a triangular embedding of K_{24}, was discovered using techniques found in Pengelley and Jungerman [5]. The ends labeled "A" and "B" are identified to form a cylindrical digraph.

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